Output Feedback Control for Uncertain Nonlinear Systems with Slowly Varying Input Delay

H. T. Dinh, N. Fischer, R. Kamalapurkar, and W. E. Dixon

***Abstract*—Output Feedback (OFB) control of a nonlinear system with time-varying actuator delay is a challenging problem because of both the the lack of full state information and the need to develop prediction of the nonlinear dynamics. In this paper, an OFB tracking controller is developed for a general second-order system with time-varying input delay, uncertainties, and additive bounded disturbances. The developed controller is a modified PD controller working in association with a predictor-like feedback term to compensate for the input delay. The PD components are formulated using the difference between a desired trajectory and an estimate of the unknown state, acquired from a dynamic neural network-based observer, to compensate for the unavailability of the true system state. A stability analysis using Lyapunov-Krasovskii functionals is provided to for uniformly ultimately bounded (UUB) tracking and UUB estimation of the unavailable state. A numerical simulation is provided to illustrate the performance of the control design and illustrate its robustness to delay variations.**

I. Introduction

For many practical systems, time delay is inevitable. For example, the torque generated by an internal combustion engine can be delayed due to fuel-air mixing, ignition delays, and cylinder pressure force propagation (see, e.g., [1], [2]). Similarly, communication delays are present in remote control applications where time is required to transmit information used for feedback control (see, e.g., master-slave teleoperation of robot in [3]–[5]). Unfortunately, time delay is a source of instability and can decrease system performance.

Various stability methods and control design techniques have been developed for systems with input delays. For nonlinear systems, Lyapunov-Krasovskii (LK) functional-based methods (cf. [6], [7]) and Lyapunov-Razumikhin methods (cf. [8], [9]) are the most widely used tools to investigate the stability of a system affected by time delays. Frequency domain methods that check for negative roots of the characteristic equation of a retarded or neutral partial differential equation [10], [11] are limited in their applicability to linear time-invariant systems with exact model knowledge. In contrast, Krasovskii-type and Razumikhin-type approaches can be applied for uncertain nonlinear systems with (time-varying) time delays. Razumikhin methods can be considered as a particular, but more conservative, case of Krasovskii methods, where Razumikhin methods can be applied to arbitrarily large, bounded time-varying delays $(0 \leq \tau(t) < \infty)$, but require input-to-state stability (ISS) of the nominal system without delay. Krasovskii methods, however, relax the ISS restriction but require a bounded derivative of the delays $(\dot{\tau}(t) \leq \varphi < 1)$.

Numerous full-state feedback controllers based on LK or Razumikhin stability criterion have been developed for nonlinear systems with input delays. Approaches in [12]–[15] provide control methods for uncertain nonlinear systems with known and unknown constant time-delays. However, time-delays are likely to vary (with respect to time or state) in practice. Several methods considering time-varying input delays in nonlinear systems have been recently investigated. Linearized controllers in [16] and [17] are only valid within a region around the linearization point. The controller developed in [18], which is an extension of [14], deals with forward complete nonlinear systems with time-varying input delay under an assumption that the plant is asymptotically stabilizable in the absence of the input delay. In [18], an invertible infinite dimensional backstepping transformation is used to yield an asymptotically stable system. An Euler-Lagrange system with a slowly varying input delay is considered in [19], where full state feedback is required.

When only the system output is available for feedback, controllers that handle both unknown system states and time-varying delays of the input are rarely investigated. Studies in [20], [21] address the output feedback (OFB) control problem for nonlinear systems with constant input delay using a linearization method. Applications with constant time delay are considered in [20], [21] where the objectives are to design OFB controllers to stabilize the systems around a set point. The controllers are first designed for delay-free linearized systems, then robustness to the delay is proven provided certain delay dependent conditions are upheld.

Challenges to design an OFB controller for uncertain nonlinear systems with time-varying input delays stem from two questions: how can we inject (partially known) negative state feedback through the delayed input, and how do we account for the delayed state which is introduced into the closed-loop system by the input. In full-state feedback methods such as [15] and [19], the first question is answered using a predictor term to provide a delay-free input to the system. The second

This research is supported in part by NSF award numbers 0901491, 1161260, and 1217908. Any opinions, findings and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the sponsoring agency.

Huyen T. Dinh is with the Department of Mechanical Engineering, University of Transport and Communications, Hanoi, Vietnam. Email: huyendtt214@gmail.com.

Nicholas Fischer, Rushikesh Kamalapurkar, and Warren E. Dixon are with the Department of Mechanical and Aerospace Engineering, University of Florida, Gainesville, FL, USA. Email: nic.r.fischer@gmail.com, {rkamalapurkar, wdixon}@ufl.edu.

question can be satisfied using an auxiliary LK functional, provided as an integral over a delay-time interval, designed such that its time derivative provides a negative feedback term that cancels delayed terms in the stability analysis. With OFB control, difficulty arises since the corresponding state is unmeasurable and can not directly be used in feedback in the closed-loop system.

In this paper, an OFB controller for an uncertain second-order nonlinear system with additive bounded disturbances is developed to compensate for the lack of full state feedback and time-varying delay of the input. The delay is assumed to be bounded and slowly varying. Motivated by our previous works in [22] and [23], a dynamic neural network (DNN)-based observer with on-line update algorithms is used to inject an estimate of an unmeasurable error signal into the closed-loop system. Then, the difference between the residual of the unmeasurable state and the unmeasurable error signal are used to design a feedback controller. Utilizing the predictor term, the delay-free residual is applied to the closed-loop system to form negative feedback of the unmeasurable state. To compensate for the the delayed cross terms resulting from the residual, an auxiliary LK functional is used, providing simultaneous uniformly ultimately bounded (UUB) estimation of the unmeasurable state and UUB tracking, despite a lack of full state feedback, time-varying input delays, model uncertainties, and exogenous disturbances. A numerical simulation using a two-link robotic manipulator is provided to illustrate the performance of the control design.

II. SYSTEM MODEL AND OBJECTIVES

Consider a control-affine second order nonlinear system of the form

$$\ddot{x} = f(x, \dot{x}) + u\left(t - \tau(t)\right) + d(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is a measurable output with a finite initial condition $x(0) = x_0$, $u\left(t - \tau(t)\right) \in \mathbb{R}^n$ represents a delayed control input, $\tau(t) \in \mathbb{R}$ is a non-negative time-varying delay, $f(x, \dot{x}) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is an unknown continuous function, and $d(t) \in \mathbb{R}^n$ is an exogenous disturbance. The subsequent development is based on the assumptions that the state $x(t)$ is measurable, the time-varying input delay $\tau(t)$ is known, and the control input vector and its past values (i.e., $u\left(t - \theta\right)\ \forall \theta \in [0,\ \tau(t)]$) are measurable. Throughout this paper, a time dependent delayed function is denoted as $\xi\left(t - \tau(t)\right)$ or ξ_τ. Additionally, the following assumptions will be exploited:

Assumption 1. The unknown function $f(x, \dot{x})$ is $\mathcal{C}^1$. **Assumption 2.** The disturbance $d(t)$ is differentiable, and $d(t), \dot{d}(t) \in \mathcal{L}_\infty$. **Assumption 3.** The time delay is bounded such that $0 \leq \tau(t) \leq \varphi_1$, and $\left|\dot{\tau}\left(t\right)\right| < \varphi_2 < \frac{1}{3}$ where $\varphi_1, \varphi_2 \in \mathbb{R}^+$ are known constants.

The universal approximation property states that given any continuous function $F : \mathbb{S} \to \mathbb{R}^n$, where $\mathbb{S}$ is a compact set, there exist ideal weights $\theta = \theta^*$, such that the output of the neural network (NN), $\hat{F}(\cdot, \theta)$, approximates $F(\cdot)$ to any arbitrary accuracy [24]. Hence, the unknown function $f(x, \dot{x})$ in (1) can be replaced by a multi-layer neural network (MLNN), and the system can be represented as

$$\ddot{x} = W^T \sigma + \varepsilon + u_\tau + d, \tag{2}$$

where $\sigma(t) \triangleq \sigma(V_1^T x(t) + V_2^T \dot{x}(t)) \in \mathbb{R}^{N+1}$ is an activation function (sigmoid, hyperbolic tangent, etc.), $W \in \mathbb{R}^{N+1 \times n}$, $V_1, V_2 \in \mathbb{R}^{n \times N}$ are unknown ideal weight matrices of the MLNN having N hidden layer neurons, and $\varepsilon(x, \dot{x}) \in \mathbb{R}^n$ is a function reconstruction error. The following assumptions will be used in the DNN-based observer and controller development and stability analysis.

Assumption 4. The ideal NN weights are bounded by known positive constants [3], i.e. $\|W\| \leq \bar{W}$, $\|V_1\| \leq \bar{V}_1$, $\|V_2\| \leq \bar{V}_2$. **Assumption 5.** The activation function $\sigma(\cdot)$ and its derivative with respect to its arguments $\sigma'(\cdot)$ are bounded [3]. This assumption is satisfied for typical activation functions (e.g., sigmoid, hyperbolic tangent). **Assumption 6.** The function reconstruction error $\varepsilon(x, \dot{x})$ is bounded [3], as $\|\varepsilon(x, \dot{x})\| \leq \bar{\varepsilon}$, where $\bar{\varepsilon}$ is known positive constant.

A contribution of this paper is the development of a continuous DNN-based observer to estimate the unmeasurable state $\dot{x}(t)$ of the input-delayed system in (1). Based on this estimate, a continuous controller is designed so that the system state $x(t)$ tracks a desired time-varying trajectory $x_d(t) \in \mathbb{R}^n$, despite uncertainties and disturbances in the system. To quantify these objectives, an estimation error $\tilde{x}(x, \hat{x}) \in \mathbb{R}^n$ and a tracking error $e(x, t) \in \mathbb{R}^n$ are defined as

$$\tilde{x} \triangleq x - \hat{x}, \quad e \triangleq x - x_d, \tag{3}$$

where $\hat{x}(t) \in \mathbb{R}^n$ is a state of the DNN observer which is introduced in the subsequent development. The desired trajectory $x_d(t)$ and its derivatives $x_d^{(i)}(t)$ $(i = 1, 2)$, are assumed to exist and be bounded. To compensate for the lack of direct measurements of $\dot{x}(t)$, a filtered estimation error $r_{es}(\tilde{x}, \dot{\tilde{x}}, \eta) \in \mathbb{R}^n$ and a filtered tracking error $r_{tr}(\dot{e}, e, e, \eta) \in \mathbb{R}^n$ are defined as

$$r_{es} \triangleq \dot{\tilde{x}} + \alpha\tilde{x} + \eta, \quad r_{tr} \triangleq \dot{e} + \alpha e + e_z + \eta, \tag{4}$$

where $\alpha \in \mathbb{R}^+$ is a positive constant gain, and $e_z(t) \in \mathbb{R}^n$ is an auxiliary time-delayed signal defined as

$$e_z \triangleq \int_{t-\tau(t)}^{t} u(\theta) d\theta. \tag{5}$$

The term $e_z(t)$ is a predictor-like term in sense that $e_z(t)$ facilitates the transformation of the delayed input into a delay-free input which can be subsequently designed. In (4), $\eta(t) \in \mathbb{R}^n$ is an output of the following dynamic filter

$$\begin{aligned} \eta &= p - (k + \alpha)\tilde{x}, \ \ \dot{\nu} = p - \alpha\nu - (k + \alpha)\tilde{x}, \\ \dot{p} &= -(k + 2\alpha)p - \nu + ((k + \alpha)^2 + 1)\tilde{x} + e, \\ p(0) &= (k + \alpha)\tilde{x}(0), \quad \nu(0) = 0, \end{aligned} \tag{6}$$

where $p(t) \in \mathbb{R}^n$ is used as an internal filter variable, $\nu(t) \in \mathbb{R}^n$ is another output of the filter, and $k \triangleq k_1 + k_2 + k_3 \in \mathbb{R}^+$, where $k_1, k_2, k_3 \in \mathbb{R}^+$ are positive constant control gain. The filtered estimation error $r_{es}(\cdot)$ and the filtered tracking error $r_{tr}(\cdot)$ are not measurable since the expressions in (4) depend on $\dot{x}(t)$.

Remark 1. The filter in (6) admits the estimation error $\tilde{x}(t)$ and the tracking error $e(\cdot)$ as its inputs and produces $\nu(t)$ and $\eta(t)$ as outputs which are related as $\eta = \dot{\nu} + \alpha\nu$, the auxiliary signal $p(t)$ is utilized to only generate the signal $\eta(t)$ without involving the unmeasurable state $\dot{x}(t)$. Hence, the filter can be physically implemented.

III. Robust DNN Observer Development

The following MLDNN architecture is proposed to observe the system in (1)

$$\ddot{\hat{x}} = \hat{W}^T\hat{\sigma} + u_\tau - (k+3\alpha)\eta, \tag{7}$$

where $\left[\hat{x}(t)^T \; \dot{\hat{x}}(t)^T\right]^T \in \mathbb{R}^{2n}$ are the states of the DNN observer, $\hat{W}(t) \in \mathbb{R}^{N+1\times n}$, $\hat{V}_1(t), \hat{V}_2(t) \in \mathbb{R}^{n\times N}$ are weight estimates, and $\hat{\sigma}(t) \triangleq \sigma(\hat{V}_1(t)^T\hat{x}(t) + \hat{V}_2(t)^T\dot{\hat{x}}(t)) \in \mathbb{R}^{N+1}$.

Remark 2. The term $(k+3\alpha)\eta(t)$ in the DNN observer (7) is a cross-term which is cancelled in the stability analysis. The MLNN term $\hat{W}(t)^T\hat{\sigma}(t)$ receives the feedback of the observer states $\hat{x}(t)$, $\dot{\hat{x}}(t)$ as its inputs, hence the observer exploits a DNN structure. Motivation for the DNN-based observer design is that the DNN can approximate nonlinear dynamic systems with any degree of accuracy [25], [26]. Moreover, the DNN includes state feedback yielding computational advantages over a static feedforward NN [27].

The weight update laws for the DNN in (7) are updated by a gradient descent update law [23] as

$$\dot{\hat{W}} = \Gamma_w proj[\hat{\sigma}(\tilde{x}+\nu)^T],\; \dot{\hat{V}}_1 = \Gamma_{v1} proj[\hat{x}(\tilde{x}+\nu)^T\hat{W}^T\hat{\sigma}'],$$
$$\dot{\hat{V}}_2 = \Gamma_{v2} proj[\dot{\hat{x}}(\tilde{x}+\nu)^T\hat{W}^T\hat{\sigma}'], \tag{8}$$

where $\Gamma_w \in \mathbb{R}^{(N+1)\times(N+1)}$, $\Gamma_{v1}, \Gamma_{v2} \in \mathbb{R}^{n\times n}$ are constant symmetric positive-definite adaptation gains, the term $\hat{\sigma}'(t)$ is defined as $\hat{\sigma}' \triangleq d\sigma(\varsigma)/d\varsigma|_{\varsigma=\hat{V}_1^T\hat{x}+\hat{V}_2^T\dot{\hat{x}}}$, and $proj(\cdot)$ is a smooth projection operator (cf. [28], [29]) used to guarantee that the weight estimates $\hat{W}(t), \hat{V}_1(t), \hat{V}_2(t)$ remain bounded.

To facilitate the subsequent analysis, (4) and (6) can be used to express the time derivative of $\eta(t)$ as

$$\dot{\eta} = -(k+\alpha)r_{es} - \alpha\eta + \tilde{x} + e - \nu. \tag{9}$$

The closed-loop dynamics of the filtered estimation error in (4) can be determined by using (2)-(4), (7) and (9) as

$$\dot{r}_{es} = W^T\sigma - \hat{W}^T\hat{\sigma} + \varepsilon + d + (k+3\alpha)\eta - (k+\alpha)r_{es}$$
$$- \alpha\eta + \tilde{x} + e - \nu + \alpha(r_{es} - \alpha\tilde{x} - \eta). \tag{10}$$

After some algebraic manipulation, the closed-loop dynamics of the filtered estimation error $r_{es}(\cdot)$ can be further expressed as

$$\dot{r}_{es} = N_1 - kr_{es} + (k+\alpha)\eta - \tilde{x}, \tag{11}$$

where the auxiliary function $N_1(e, \tilde{x}, \nu, r_{es}, r_{tr}, e_z, \hat{W}, \hat{V}_1, \hat{V}_2, t) \in \mathbb{R}^n$ is $N_1 \triangleq W^T\sigma - \hat{W}^T\hat{\sigma} + \varepsilon + d - (\alpha^2-2)\tilde{x} - \nu + e$. Using (3), (4), Assumptions 2, 4-6, the $proj(\cdot)$ algorithm in (8), and the Mean Value Theorem, the auxiliary function $N_1(\cdot)$ can be upper-bounded as

$$\|N_1\| \le \zeta_1\|z\| + \zeta_2, \tag{12}$$

where $\zeta_1, \zeta_2 \in \mathbb{R}^+$ are computable positive constants, and $z(\tilde{x}, e, r_{es}, r_{tr}, \nu, \eta, e_z) \in \mathbb{R}^{7n}$ is defined as $z \triangleq [\tilde{x}^T\; e^T\; r_{es}^T\; r_{tr}^T\; \nu^T\; \eta^T\; e_z^T]^T$.

IV. Robust Tracking Control Development

The control objective is to force the system state to track the desired trajectory, $x_d(t)$, despite uncertainties, disturbances, and time-delays in the system. Quantitatively, this objective is to regulate the tracking error $e(t)$ to zero. Using (2)-(6) and (9), the open-loop dynamics of the filtered tracking error in (4) can be expressed as

$$\dot{r}_{tr} = W^T\sigma + u_\tau + \varepsilon + d - \ddot{x}_d + \alpha(r_{tr} - \alpha e - \eta - e_z)$$
$$+ \dot{e}_z - (k+\alpha)r_{es} - \alpha\eta + \tilde{x} + e - \nu. \tag{13}$$

Based on the error system formulation in (5), the time derivative of $e_z(t)$ can be calculated as $\dot{e}_z = u - u_\tau + u_\tau\dot{\tau}$. Hence, the open-loop error system in (13) contains a delay-free control input. Based on (13), the derivative of $e_z(t)$, and the subsequent stability analysis, the control input is designed as

$$u = -(k+\alpha)(\dot{\hat{e}} + \alpha\hat{e} + e_z), \tag{14}$$

where the tracking error estimate $\hat{e}(\hat{x}, t) \in \mathbb{R}^n$ is defined as $\hat{e} \triangleq \hat{x} - x_d$. Based on the fact that the estimated states $\hat{x}(t)$ and $\dot{\hat{x}}(t)$ are measurable, the tracking error estimate $\hat{e}(\cdot)$ and its derivative $\dot{\hat{e}}(\cdot)$ are measurable; moreover, the filtered tracking error $r_{tr}(\cdot)$ is related to the filtered estimation error $r_{es}(\cdot)$ via the tracking error estimate $\hat{e}(\cdot)$, and the control input $u(t)$ is related to $r_{es}(\cdot)$, $r_{tr}(\cdot)$ as below

$$r_{tr} = r_{es} + \dot{\hat{e}} + \alpha\hat{e} + e_z,\;\; u(t) = (k+\alpha)(r_{es} - r_{tr}). \tag{15}$$

The relation in (15) shows that even though both the filtered tracking error $r_{tr}(\cdot)$ and the filtered estimation error $r_{es}(\cdot)$ are unmeasurable, the difference between $r_{tr}(\cdot)$ and $r_{es}(\cdot)$ is measurable. The DNN observer provides negative feedback of the filtered estimation error $r_{es}(\cdot)$ to guarantee the convergence of the estimated states, and the controller in (14) compensates for the difference between $r_{es}(\cdot)$ and $r_{tr}(\cdot)$ to obtain negative feedback of the filtered tracking error $r_{tr}(\cdot)$, allowing for convergence of the tracking error to be achieved.

Using (13)-(15), the closed-loop error system becomes

$$\dot{r}_{tr} = N_2 - kr_{tr} + u_\tau\dot{\tau} - e, \tag{16}$$

where the auxiliary function $N_2(e, \tilde{x}, \eta, \nu, e_z, r_{tr}, t) \in \mathbb{R}^n$ is defined $N_2 \triangleq W^T\sigma - (\alpha^2-2)e - \nu + \tilde{x} - 2\alpha\eta - \alpha e_z + \varepsilon + d - \ddot{x}_d$. Similarly, using (3), (4), Assumptions 4-6, and the $proj(\cdot)$ algorithm in (8), the auxiliary function $N_2(\cdot)$ can be upper-bounded as

$$\|N_2\| \le \zeta_3\|z\| + \zeta_4, \tag{17}$$

where $\zeta_3, \zeta_4 \in \mathbb{R}^+$ are computable positive constants.

To facilitate the subsequent stability analysis, let $y(z,P,Q) \in \mathbb{R}^{7n+2}$ be defined as $y \triangleq [z^T \sqrt{P} \sqrt{Q}]^T$, where $P(u,t,\tau)$, $Q(r_{es}, r_{tr}, t, \tau) \in \mathbb{R}$ denote positive-definite LK functionals defined as

$$P \triangleq \omega \int_{t-\tau}^{t} \left(\int_{s}^{t} \|u(\theta)\|^2 d\theta \right) ds, \tag{18}$$

$$Q \triangleq \frac{k+\alpha}{4} \int_{t-\tau}^{t} \|r_{es}(\theta) - r_{tr}(\theta)\|^2 d\theta, \tag{19}$$

and $\omega \in \mathbb{R}^+$ is an adjustable constant.

V. Lyapunov Stability Analysis For DNN-based Observation and Control

Theorem 1. *The DNN-based observer and controller proposed in (7) and (14), along with the weight update laws in (8) ensure uniformly ultimately bounded estimation and tracking in sense that*

$$\left\|\dot{\tilde{x}}(t)\right\| \leq \epsilon_1 \exp(-\epsilon_2 t) + \epsilon_3, \ \|e(t)\| \leq \epsilon_4 \exp(-\epsilon_5 t) + \epsilon_6, \tag{20}$$

where $\epsilon_i \in \mathbb{R}^+$, $\forall i = 1, 2, ..., 6$ are known constants, provided $\tau(t)$ and $\dot{\tau}(t)$ are sufficiently small, and the following sufficient condition is satisfied

$$\frac{1}{2} \min \left[\inf_{\tau,\dot{\tau}} \left(k_1 - (k+\alpha) \left(\frac{|\dot{\tau}|+1}{2} + 2\omega\tau(k+\alpha) \right) \right), \alpha - \frac{\psi^2}{2}, \inf_{\tau,\dot{\tau}} \left(\frac{\omega(1-\dot{\tau})}{2\tau} - \frac{1}{2\psi^2} \right) \right] > \frac{(\zeta_1^2 + \zeta_3^2)}{4k_2}, \tag{21}$$

where $\psi \in \mathbb{R}^+$ is a known, adjustable constant.

Proof: Consider the Lipschitz continuous, positive-definite, Lyapunov functional candidate, $V_L(y,t) : \mathbb{R}^{7n+2} \times [0,\infty) \to \mathbb{R}$, defined as

$$V_L \triangleq \frac{1}{2}\tilde{x}^T\tilde{x} + \frac{1}{2}e^Te + \frac{1}{2}\nu^T\nu + \frac{1}{2}\eta^T\eta + \frac{1}{2}r_{es}^T r_{es} + \frac{1}{2}r_{tr}^T r_{tr} + P + Q, \tag{22}$$

which satisfies the following inequalities: $U_1(y) \leq V_L(y,t) \leq U_2(y)$, where $U_1(y), U_2(y) \in \mathbb{R}$ are continuous positive-definite functions defined as $U_1 \triangleq \frac{1}{2}\|y\|^2$, $U_2 \triangleq \|y\|^2$. Using (4), (5), (9), (11), and (16), and by applying the Leibniz Rule to determine the time derivative of (18) and (19), the time derivative of (22) can be calculated as

$$\begin{aligned} \dot{V}_L =& \tilde{x}^T(r_{es} - \alpha\tilde{x} - \eta) + e^T(r_{tr} - \alpha e - \eta - e_z) \\ &+ \eta^T(-(k+\alpha)r_{es} - \alpha\eta + \tilde{x} + e - \nu) \\ &+ \nu^T(\eta - \alpha\nu) + r_{es}^T(N_1 - kr_{es} + (k+\alpha)\eta - \tilde{x}) \\ &+ r_{tr}^T(N_2 - kr_{tr} + u_\tau\dot{\tau} - e) + \omega\tau\|u\|^2 \\ &- \omega(1-\dot{\tau})\int_{t-\tau}^{t}\|u(\theta)\|^2 d\theta + \frac{k+\alpha}{4}\|r_{es} - r_{tr}\|^2 \\ &- (1-\dot{\tau})\frac{k+\alpha}{4}\|r_{es_\tau} - r_{tr_\tau}\|^2. \end{aligned} \tag{23}$$

Canceling common terms, and utilizing the relationship in (15), the expression in (23) can be regrouped as

$$\begin{aligned} \dot{V}_L =& -\alpha\tilde{x}^T\tilde{x} - \alpha e^Te - e^Te_z - \alpha\nu^T\nu - \alpha\eta^T\eta - kr_{es}^Tr_{es} \\ &- kr_{tr}^Tr_{tr} + \dot{\tau}(k+\alpha)r_{tr}^T(r_{es_\tau} - r_{tr_\tau}) + r_{es}^TN_1 \\ &+ r_{tr}^TN_2 + \omega\tau(k+\alpha)^2\|r_{es} - r_{tr}\|^2 \\ &+ \frac{k+\alpha}{4}\|r_{es} - r_{tr}\|^2 - \omega(1-\dot{\tau})\int_{t-\tau}^{t}\|u(\theta)\|^2 d\theta \\ &- (1-\dot{\tau})\frac{k+\alpha}{4}\|r_{es_\tau} - r_{tr_\tau}\|^2. \end{aligned} \tag{24}$$

Young's inequality can be used to upper bound select terms in (24) as

$$\begin{aligned} \|e\|\|e_z\| &\leq \frac{\psi^2}{2}\|e\|^2 + \frac{1}{2\psi^2}\|e_z\|^2, \\ \|r_{tr}\|\|r_{es_\tau} - r_{tr_\tau}\| &\leq \frac{1}{2}\|r_{tr}\|^2 + \frac{1}{2}\|r_{es_\tau} - r_{tr_\tau}\|^2, \\ \|r_{es} - r_{tr}\|^2 &\leq 2\|r_{tr}\|^2 + 2\|r_{es}\|^2. \end{aligned} \tag{25}$$

Using (15), (25), (24) can be upper bounded as

$$\begin{aligned} \dot{V}_L \leq& -\alpha\|\tilde{x}\|^2 - \left(\alpha - \frac{\psi^2}{2}\right)\|e\|^2 + \frac{1}{2\psi^2}\|e_z\|^2 \\ &- \alpha\|\nu\|^2 - \alpha\|\eta\|^2 - k\|r_{es}\|^2 - k\|r_{tr}\|^2 \\ &+ \frac{|\dot{\tau}|(k+\alpha)}{2}\left(\|r_{tr}\|^2 + \|r_{es_\tau} - r_{tr_\tau}\|^2\right) \\ &- \omega(1-\dot{\tau})\int_{t-\tau}^{t}\|u(\theta)\|^2 d\theta + \|r_{es}\|\|N_1\| \\ &+ \left(2\omega\tau(k+\alpha)^2 + \frac{k+\alpha}{2}\right)\left(\|r_{tr}\|^2 + \|r_{es}\|^2\right) \\ &- (1-\dot{\tau})\frac{k+\alpha}{4}\|r_{es_\tau} - r_{tr_\tau}\|^2 + \|r_{tr}\|\|N_2\|. \end{aligned} \tag{26}$$

Using the Cauchy-Schwartz inequality and (5), the integral term in (26) can be upper bounded as

$$-\omega(1-\dot{\tau})\int_{t-\tau}^{t}\|u(\theta)\|^2 d\theta \leq -\frac{\omega(1-\dot{\tau})}{2\tau}\|e_z\|^2 - \frac{\omega(1-\dot{\tau})}{2}\int_{t-\tau}^{t}\|u(\theta)\|^2 d\theta,$$

and based on Assumption 3, $\frac{|\dot{\tau}|(k+\alpha)}{2} \leq \frac{(1-\dot{\tau})(k+\alpha)}{4}$. Utilizing (12), (17), and above inequalities, (26) can be rewritten as

$$\begin{aligned} \dot{V}_L \leq& -\alpha\|\tilde{x}\|^2 - \left(\alpha - \frac{\psi^2}{2}\right)\|e\|^2 - \alpha\|\nu\|^2 - \alpha\|\eta\|^2 \\ &- \left(\frac{\omega(1-\dot{\tau})}{2\tau} - \frac{1}{2\psi^2}\right)\|e_z\|^2 - k\|r_{es}\|^2 - k\|r_{tr}\|^2 \\ &+ (\zeta_1\|z\| + \zeta_2)\|r_{es}\| + (\zeta_3\|z\| + \zeta_4)\|r_{tr}\| \\ &+ \frac{|\dot{\tau}|(k+\alpha)}{2}\|r_{tr}\|^2 - \frac{\omega(1-\dot{\tau})}{2}\int_{t-\tau}^{t}\|u(\theta)\|^2 d\theta \\ &+ \left(2\omega\tau(k+\alpha)^2 + \frac{k+\alpha}{2}\right)\left(\|r_{tr}\|^2 + \|r_{es}\|^2\right). \end{aligned} \tag{27}$$

Provided the sufficient condition in (21) is satisfied, then the auxiliary constant $\beta \in \mathbb{R}^+$ can be defined as

$$\beta \triangleq \frac{1}{2} min \left[\alpha - \frac{\psi^2}{2}, \inf_{\tau,\dot{\tau}} \left(\frac{\omega(1-\dot{\tau})}{2\tau} - \frac{1}{2\psi^2} \right), \inf_{\tau,\dot{\tau}} \left(k_1 - (k+\alpha) \left(\frac{|\dot{\tau}|+1}{2} + 2\omega\tau(k+\alpha) \right) \right) \right], \quad (28)$$

where $k_1 \in \mathbb{R}^+$ is introduced in (6). Note that provided τ is sufficiently small, k_1 and α can be made large enough to ensure $\beta > 0$. Using (28) and by completing the squares, the expression in (27) can be further upper bounded as

$$\dot{V}_L \leq -2\beta \|z\|^2 + \frac{\left(\zeta_1^2 + \zeta_3^2\right)\|z\|^2}{4k_2} + \frac{\zeta_2^2 + \zeta_4^2}{4k_3} - \frac{\omega(1-\dot{\tau})}{2} \int_{t-\tau}^{t} \|u(\theta)\|^2 d\theta. \quad (29)$$

Using the inequality [15]

$$\int_{t-\tau}^{t} \left(\int_{s}^{t} \|u(\theta)\|^2 d\theta \right) ds \leq \tau \sup_{s\in[t-\tau,t]} \left[\int_{s}^{t} \|u(\theta)\|^2 d\theta \right] = \tau \int_{t-\tau}^{t} \|u(\theta)\|^2 d\theta,$$

the expression in (29) can be upper bounded as

$$\dot{V}_L \leq -\left(\beta - \frac{\zeta_1^2+\zeta_3^2}{4k_2} \right) \|z\|^2 - \frac{\omega(1-\dot{\tau})}{4} \int_{t-\tau}^{t} \|u(\theta)\|^2 d\theta + \lambda - \frac{\omega(1-\dot{\tau})}{4\tau} \int_{t-\tau}^{t} \left(\int_{s}^{t} \|u(\theta)\|^2 d\theta \right) ds. \quad (30)$$

By further utilizing (18), (19) and (15), the inequality in (30) can be written as

$$\dot{V}_L \leq -\left(\beta - \frac{\zeta_1^2+\zeta_3^2}{4k_2} \right) \|z\|^2 + \lambda - \omega(1-\dot{\tau})(k+\alpha)Q - \frac{(1-\dot{\tau})}{4\tau} P. \quad (31)$$

Provided the sufficient condition in (21) is satisfied, then using the definition of $y(\cdot)$, and (22), the inequality in (31) can be written as $\dot{V}_L \leq -\beta_2 V_L + \lambda$, where $\beta_2, \lambda \in \mathbb{R}^+$ are given by $\beta_2 = min \left[\left(\beta - \frac{\zeta_1^2+\zeta_3^2}{4k_2} \right), \underset{\tau,\dot{\tau}}{inf} \left(\frac{1-\dot{\tau}}{4\tau} \right), \underset{\tau,\dot{\tau}}{inf} \left(\omega(1-\dot{\tau})(k+\alpha) \right) \right]$, and $\lambda = \frac{\zeta_2^2+\zeta_4^2}{4k_3}$. Note that provided τ is sufficiently small, k_1 and α can be made large enough to ensure $\beta_2 > 0$. The linear differential inequality of $V_L(\cdot)$ can be solved as

$$V_L(y,t) \leq e^{-\beta_2 t} V_L(0) + \beta_2^{-1} \lambda \left[1 - e^{-\beta_2 t} \right], \ \forall t \geq 0. \quad (32)$$

Based on (22) and (32), it can be concluded that $e(\cdot)$, $\tilde{x}(\cdot)$, $r_{es}(\cdot)$, $r_{tr}(\cdot)$, $\eta(t)$, $\nu(t) \in \mathcal{L}_\infty$, hence using the definition of $r_{es}(\cdot)$ in (4), there exist known constants ϵ_i, $i = 1, 2, \ldots 6$ such that $\|e(t)\|$ and $\left\|\dot{\hat{x}}(t)\right\|$ are bounded in sense of (20). Utilizing the relation between $u(t)$, $r_{es}(\cdot)$ and $r_{tr}(\cdot)$ in (15), $u(t) \in \mathcal{L}_\infty$. ■

TABLE I
LINK1 AND LINK 2 RMS TRACKING ERRORS AND RMS ESTIMATION ERRORS.

	RMS Tracking (deg)		RMS Estimation (deg/s)	
Time-Delay $\tau(t)$ (ms)	Link1	Link2	Link 1	Link2
$2sin\left(\frac{t}{10}\right)+5$	0.1346	0.1820	0.0366	0.1257
$2sin\left(\frac{t}{10}\right)+10$	0.2278	0.3024	0.0491	0.2050
$5sin\left(\frac{t}{2}\right)+10$	0.2351	0.3463	0.0546	0.2350
$5sin\left(\frac{t}{2}\right)+20$	0.9134	0.9350	0.2097	0.6255

TABLE II
RMS ERRORS FOR CASES OF UNCERTAINTY IN TIME-VARYING DELAY SEEN BY THE PLANT AS COMPARED TO THE DELAY OF THE CONTROLLER.

	RMS Tracking (deg)		RMS Estimation (deg/s)	
Time-Delay Variance	Link1	Link2	Link 1	Link2
-30% magnitude	0.2237	0.2567	0.0470	0.1722
-10% magnitude	0.2278	0.2710	0.0491	0.1818
0% magnitude	0.2278	0.3024	0.0491	0.2050
10% magnitude	0.2322	0.3040	0.0516	0.2053
30% magnitude	0.2342	0.3119	0.0531	0.2105
10% offset	0.2429	0.3489	0.0577	0.2350
30% offset	0.2783	0.4815	0.0705	0.3288
10% magnitude, 10% offset	0.2641	0.3023	0.0626	0.2021

VI. SIMULATION RESULTS

The controller in (14) was simulated using two-link planar robot manipulator dynamics to examine the performance and robustness to variations in the input delay. Motivation for using robot dynamics stems from the fact that the dynamics can be expressed as an Euler-Lagrange system that is common to a large class of practical engineering systems. The dynamics used in the simulation are $M(q)\ddot{q} + V_m(q,\dot{q})\dot{q} + F(\dot{q}) + \tau_d = \tau_u(t-\tau(t))$, where $M(q) \triangleq \begin{bmatrix} p_1 + 2p_3c_2 & p_2 + p_3c_2 \\ p_2 + p_3c_2 & p_2 \end{bmatrix} \in \mathbb{R}^{2\times2}$ denotes an inertia matrix, $V_m(q,\dot{q}) \triangleq \begin{bmatrix} -p_3s_2\dot{q}_2 & -p_3s_2(\dot{q}_1+\dot{q}_2) \\ p_3s_2\dot{q}_1 & 0 \end{bmatrix} \in \mathbb{R}^{2\times2}$ denotes an unknown centripetal-Coriolis matrix, $F(\dot{q}) \triangleq \begin{bmatrix} f_{d1} & 0 \\ 0 & f_{d2} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \in \mathbb{R}^2$ is a friction, $\tau_d(t) \in \mathbb{R}^2$ is an unknown external disturbance, $q(t), \dot{q}(t), \ddot{q}(t) \in \mathbb{R}^2$ denote the joint position, velocity, and acceleration, and $\tau_u(t-\tau(t)) \in \mathbb{R}^2$ is the delayed control torque, where $p_1 = 3.473\, kg\cdot m^2$, $p_2 = 0.196\, kg\cdot m^2$, $p_3 = 0.242\, kg\cdot m^2$, $f_{d1} = 5.3\ Nm\ s$, $f_{d2} = 1{,}1\ Nm\ s$, $c_2 \triangleq cos(q_2)$, and $s_2 \triangleq sin(q_2)$. The dynamics can be rewritten in the form of (1) where $x(t) = \begin{bmatrix} q_1(t), & q_2(t) \end{bmatrix}^T$, $f(x,\dot{x},t) \triangleq -M^{-1}(q)(V_m(q,\dot{q})\dot{q} + F(\dot{q}))$, $u_\tau \triangleq M^{-1}(q)\tau_u(t-\tau)$, and $d(x,t) \triangleq -M^{-1}(q)\tau_d(t)$. An additive non-vanishing exogenous disturbance was applied as $\tau_{d1} = 0.2sin\left(\frac{t}{2}\right)$, and $\tau_{d2} = 0.1sin\left(\frac{t}{4}\right)$. The desired trajectories for Links 1 and 2 for all simulations are selected as $x_{d1}(t) = 2x_{d2}(t) = 40sin(1.5t)\left(1 - e^{-0.01t^3}\right)\ deg$. The initial conditions of the system and the observer are chosen as $x(t) = \dot{x}(t) = \hat{x}(t) = \dot{\hat{x}}(t) = [0,0]^T$. In the simulation, we assume that the inertia matrix $M(q)$ is known. To illustrate the performance of the developed method, simulations are executed using various time-varying delays. The time delays are selected as sinusoidal functions with various magnitudes, frequencies and displacement offsets. For each case, Link 1 and Link 2 RMS tracking errors and RMS estimation errors are shown in Table I. The results clearly show that the system performance is better with small and slowly varying time-delays.

In the theoretical analysis, the time-delay is assumed to be exactly known. However, to examine the robustness of the developed controller with respect to the time-delay parameters, the input delay entering the plant is varied from the delay used in the controller feedback. The feedback delay of the controller is selected as $\tau(t) = 2sin\left(\frac{t}{10}\right) + 10\,ms$. Table II presents simulation results where the magnitude and/or offset of plant delay are varied from the corresponding parameters of the controller delay. The results suggest that the controller is robust to variances in delay magnitude and offset. However, smaller variances result in better performance. The developed approach has been proven for exact knowledge of the time delay, but simulation results also suggest some robustness with regard to uncertainties in the time delay. Future studies will consider the development of OFB controllers for uncertain nonlinear systems with unknown time-varying input delays.

VII. Conclusion

A continuous OFB controller is developed for uncertain second-order nonlinear systems affected by time-varying input delays and additive bounded disturbances. The delay is assumed to be bounded and slowly varying. A DNN-based observer works in junction with the controller to provide an estimate of the unmeasurable state. A Lyapunov-based stability analysis is used to prove simultaneous UUB estimation of the unmeasurable state and UUB tracking in the presence of model uncertainties, disturbances and time delays.

References

[1] J. A. Cook and B. K. Powell, "Modeling of an internal combustion engine for control analysis," *IEEE Control Syst. Mag*, pp. 20–25, 1988.

[2] M. Kao and J. J. Moskwa, "Turbocharged diesel engine modeling for nonlinear engine control and state estimation," *ASME J. Dyn. Syst., Meas., Control*, vol. 117, pp. 20–30, 1995.

[3] F. L. Lewis, R. Selmic, and J. Campos, *Neuro-Fuzzy Control of Industrial Systems with Actuator Nonlinearities*. Philadelphia, PA, USA: Society for Industrial and Applied Mathematics, 2002.

[4] G. Niemeyer and J.-J. Slotine, "Telemanipulation with time delays," *Int. J. Robot. Res.*, vol. 23, pp. 873–890, 2004.

[5] F. Hashemzadeh, I. Hassanzadeh, M. Tavakoli, and G. Alizadeh, "Adaptive control for state synchronization of nonlinear haptic telerobotic systems with asymmetric varying time delays," *J. Intell. Robot. Syst.*, 2012.

[6] X. Li and C. de Souza, "Delay-dependent robust stability and stabilization of uncertain linear delay systems: a linear matrix inequality approach," *IEEE Trans. Autom. Control*, vol. 42, no. 8, pp. 1144–1148, Aug 1997.

[7] V. B. Kolmanovskii, S.-I. Niculescu, and J.-P. Richard, "On the liapunov-krasovskii functionals for stability analysis of linear delay systems," *Int. J. Control*, vol. 72, pp. 374 – 384, 1999.

[8] B. S. Razumikhin, "Application of liapunov's method to problems in the stability of systems with a delay," *Automat. i Telemeh*, vol. 21, pp. 740–749, 1960.

[9] M. Jankovic, "Control Lyapunov-Razumikhin functions and robust stabilization of time delay systems," *IEEE Trans. Autom. Control*, vol. 46, no. 7, pp. 1048–1060, 2001.

[10] R. Bellman and K. L. Cooke, *Differential- Difference Equations*. Academic Press, 1963.

[11] L. E. Elsgolts and S. B. Norkin, *Introduction to the theory and application of differential equations with deviating arguments*. Academic Press, 1973, vol. 105.

[12] F. Mazenc, S. Mondie, R. Francisco, P. Conge, I. Lorraine, and F. Metz, "Global asymptotic stabilization of feedforward systems with delay in the input," *IEEE Trans. Autom. Control*, vol. 49, (5), pp. 844–850, 2004.

[13] B. Chen, X. Liu, and S. Tong, "Robust fuzzy control of nonlinear systems with input delay," *Chaos, Solitons & Fractals*, vol. 37, no. 3, pp. 894–901, 2008.

[14] M. Krstic, "Input delay compensation for forward complete and strict-feedforward nonlinear systems," *IEEE Trans. Autom. Control*, vol. 55, pp. 287–303, feb. 2010.

[15] N. Sharma, S. Bhasin, Q. Wang, and W. E. Dixon, "Predictor-based control for an uncertain Euler-Lagrange system with input delay," *Automatica*, vol. 47, no. 11, pp. 2332–2342, 2011.

[16] L. Guo, "H infin; output feedback control for delay systems with nonlinear and parametric uncertainties," *Proc. IEE Control Theory Appl.*, vol. 149, no. 3, pp. 226 –236, 2002.

[17] W. Li, Y. Dong, and X. Wang, "Robust h infin; control of uncertain nonlinear systems with state and input time-varying delays," in *Proc. Chin. Control Decis. Conf.*, 2010, pp. 317–321.

[18] N. Bekiaris-Liberis and M. Krstic, "Compensation of time-varying input delay for nonlinear systems," in *Mediterr. Conf. Control and Autom.*, Corfu, Greece, 2011.

[19] N. Fischer, R. Kamalapurkar, N. Fitz-Coy, and W. E. Dixon, "Lyapunov-based control of an uncertain Euler-Lagrange system with time-varying input delay," in *American Control Conference*, Montréal, Canada, June 2012, pp. 3919–3924.

[20] A. Ailon, R. Segev, and S. Arogeti, "A simple velocity-free controller for attitude regulation of a spacecraft with delayed feedback," *IEEE Trans. Autom. Control*, vol. 49, no. 1, pp. 125–130, 2004.

[21] A. Ailon, "Asymptotic stability in a flexible-joint robot with model uncertainty and multiple time delays in feedback," *J. Franklin Inst.*, vol. 341, no. 6, pp. 519–531, 2004.

[22] H. T. Dinh, R. Kamalapurkar, S. Bhasin, and W. E. Dixon, "Dynamic neural network-based robust observers for second-order uncertain nonlinear systems," in *Proc. IEEE Conf. Decis. Control*, Orlando, FL, 2011, pp. 7543–7548.

[23] H. T. Dinh, S. Bhasin, D. Kim, and W. E. Dixon, "Dynamic neural network-based global output feedback tracking control for uncertain second-order nonlinear systems," in *American Control Conference*, Montréal, Canada, June 2012, pp. 6418–6423.

[24] K. Hornick, "Approximation capabilities of multilayer feedforward networks," *Neural Netw.*, vol. 4, pp. 251–257, 1991.

[25] M. Polycarpou and P. Ioannou, "Identification and control of nonlinear systems using neural network models: Design and stability analysis," *Systems Report 91-09-01, University of Southern California*, 1991.

[26] K. Funahashi and Y. Nakamura, "Approximation of dynamic systems by continuous-time recurrent neural networks," *Neural Netw.*, vol. 6, pp. 801–806, 1993.

[27] M. Gupta, L. Jin, and N. Homma, *Static and dynamic neural networks: from fundamentals to advanced theory*. Wiley-IEEE Press, 2003.

[28] W. E. Dixon, A. Behal, D. M. Dawson, and S. Nagarkatti, *Nonlinear Control of Engineering Systems: A Lyapunov-Based Approach*. Birkhauser: Boston, 2003.

[29] M. Krstic, P. V. Kokotovic, and I. Kanellakopoulos, *Nonlinear and Adaptive Control Design*. John Wiley & Sons, 1995.