Dynamic Neural Network-based Robust Observers for Uncertain Nonlinear Systems

H. T. Dinh, R. Kamalapurkar, S. Bhasin, and W. E. Dixon

***Abstract*—A dynamic neural network (DNN) based robust observer for uncertain nonlinear systems is developed. The observer structure consists of a DNN to estimate the system dynamics on-line, a dynamic filter to estimate the unmeasurable state and a sliding mode feedback term to account for modeling errors and exogenous disturbances. The observed states are proven to asymptotically converge to the system states of high-order uncertain nonlinear systems through Lyapunov-based analysis. Simulations and experiments on a two-link robot manipulator are performed to show the effectiveness of the proposed method in comparison to several other state estimation methods.**

I. Introduction

Full state feedback is not available in many practical systems. In the absence of sensors, the requirement of full-state feedback for the controller is typically fulfilled by using $ad\ hoc$ numerical differentiation techniques, which are sensitive to noise, leading to unusable state estimates. Observers are an alternative method to numerical methods. Several nonlinear observers are available in literature to estimate unmeasurable states. For instance, sliding mode observers were designed for nonlinear systems in [1]–[3] based on an assumption that exact model knowledge of the dynamics is available. Model-based observers are also developed in [4] and [5] which require a high-gain to guarantee estimation error regulation. The observers introduced in [6] and [7] are both applied for Lagrangian dynamic systems to estimate the velocity. Global exponential convergence to the true velocity is obtained in [6], and a global asymptotic result is proven in [7]. The result in [6] is based on the immersion and invariance (I&I) approach to reconstruct the unmeasurable state. The use of this approach requires the solution of a partial differential equation (PDE). Given the challenge of finding such a solution, an approximation technique is employed that introduces error in the estimation, the effects of which are dominated by high-gain terms introduced in the observer dynamics. In [7], the system dynamics must be expressed in a non-minimal model and feedback from force sensors are used to develop a velocity estimate. In [8], a constrained optimal observer is developed for a nonlinear system under the assumption of exact model knowledge, where a nonquadratic performance cost function is used to impose magnitude constraints on an observer gain matrix.

The design of robust observers for uncertain nonlinear systems is considered in [9]–[12]. In [9], a second-order sliding mode observer for uncertain systems using a super-twisting algorithm is developed, where a nominal model of the system is assumed to be available and estimation errors are proven to converge in finite-time to a bounded set around the origin. In [10], the developed observer guarantees that the state estimates exponentially converge to the actual state, if there exists a vector function satisfying a complex set of matching conditions. An asymptotic velocity observer is developed in [11] for general second-order systems; however, all nonlinear uncertainties in the system are damped out by a sliding-mode term resulting in high-frequency state estimates. In [12], a high-gain derivative estimator is developed to estimate the derivative(s) of a signal in the presence of measurement noise. In the absence of noise, the derivative estimation error asymptotically converges as the observer gain grows to infinity.

Neural networks (NN) and fuzzy logic systems provide an effective approximation method that facilitates new observer designs, improving and complementing the base of conventional observer design approaches. For example, the approachs in [13]–[17] use the universal approximation property in adaptive observer designs. However, estimation errors in [13]–[17] are only guaranteed to be bounded due to function reconstruction errors resulting from the NN or fuzzy system.

The challenge to obtain asymptotic estimation stems from the fact that to robustly account for disturbances, feedback of the unmeasurable error and its estimate is required. Typically, feedback of the unmeasurable error is obtained by taking the derivative of the measurable state and manipulating the resulting dynamics (e.g., this is the approach used in methods such as [11] and [13]). However, such an approach provides a linear feedback term of the unmeasurable state. Hence, a sliding mode term could not be simply added to the NN structure of the result in [13], for example, to yield an asymptotic result, because it would require the signum of the unmeasurable state. It is unclear how such a nonlinear function of the unmeasurable state can be injected in the closed-loop error system using traditional methods. Likewise, it is not clear how to simply add a NN-based feedforward estimation of the nonlinearities in results such

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as [11] because of the need to inject nonlinear functions of the unmeasurable state.

The approach used in this paper circumvents the challenge of injecting feedback to yield as asymptotic result by using nonlinear (sliding-mode) feedback of the measurable state, and then exploiting the recurrent nature of a dynamic neural network (DNN) structure to inject terms that cancel cross terms associated with the unmeasurable state. The approach is facilitated by using the filter structure inspired by [11] and a novel stability analysis. The stability analysis is based on the idea of segregating the nonlinear uncertainties into terms which can be upper-bounded by constants and terms which can upper-bounded by states. The terms upper-bounded by states can be canceled by the linear feedback of the measurable errors, while the terms upper-bounded by constants are partially rejected by the sliding mode feedback (of the measurable state) and partially eliminated by the novel DNN-based weight update laws.

The contribution of this paper (and its preliminary version in [18]) is that the observer is designed for N^{th} order uncertain nonlinear systems, where the output of the N^{th} order system is assumed to be measurable up to $N-1$ derivatives. The on-line approximation of the unmeasurable uncertain nonlinearities via the DNN structure should heuristically improve the performance of methods that only use high-gain feedback. Asymptotic convergence of the estimated states to the real states is proven using a Lyapunov-based analysis. The developed observer can be used separately from the controller even if the relative degree between the control input and the output is arbitrary. Simulation and experiment results on a two-link robot manipulator, a second-order system, indicate the effectiveness of the proposed observer when compared with the standard numerical central differentiation algorithm, along with the high-gain observer proposed in [12] and the observer in [11].

II. DNN-based Observer Development

Consider an N^{th} order control affine nonlinear system given in MIMO Brunovsky form as

$$\begin{aligned}\dot{x}_1 &= x_2,\\ &\vdots\\ \dot{x}_{N-1} &= x_N,\\ \dot{x}_N &= f(x)+G(x)u+d,\\ y &= x_1,\end{aligned} \tag{1}$$

where $x=\left[x_1^T\ x_2^T\ \ldots x_N^T\right]^T\in\mathbb{R}^{Nn}$ is the generalized state of the system, $u\in\mathbb{R}^m$ is the control input, $f:\mathbb{R}^{Nn}\to\mathbb{R}^n$, $G:\mathbb{R}^{Nn}\to\mathbb{R}^{n\times m}$ are unknown continuous functions, $d\in\mathbb{R}^n$ is an external disturbance, the system output $y\in\mathbb{R}^n$ is measurable up to $N-1^{th}$ derivatives, i.e. $x_i, i=1,2..,N-1$ are measurable and x_N is unmeasurable with the finite initial condition $y(0)=y_0$. The following assumptions about the system in (1) will be utilized in the observer development.

Assumption 1. The state x is bounded, i.e, $x_i\in\mathcal{L}_\infty$, $i=1,2..,N$, and is partially unmeasurable, i.e, only x_N is unmeasurable.

Assumption 2. The unknown functions f and G, and the control input u is C^1, and $u,\dot{u}\in\mathcal{L}_\infty$.

Assumption 3. The disturbance d is differentiable, and $d,\dot{d}\in\mathcal{L}_\infty$.

The universal approximation property states that given any continuous function $F:\mathbb{S}\to\mathbb{R}^n$, where $\mathbb{S}$ is a compact set, there exist ideal weights $\theta=\theta^*$, such that the output of the NN, $\hat{F}(\cdot,\theta)$ approximates $F(\cdot)$ to an arbitrary accuracy [19]. Hence, the unknown functions f and G in (1) can be replaced by multi-layer NNs (MLNN) as

$$\begin{aligned}f(x) &= W_f^T\sigma_f(\sum_{j=1}^{N}V_{f_j}^T x_j)+\varepsilon_f(x),\\ g_i(x) &= W_{gi}^T\sigma_{gi}(\sum_{j=1}^{N}V_{gi_j}^T x_j)+\varepsilon_{gi}(x),\end{aligned} \tag{2}$$

where $W_f\in\mathbb{R}^{L_f+1\times n}$, $V_{f_j}\in\mathbb{R}^{n\times L_f}$ are unknown ideal weight matrices of the MLNN having L_f hidden layer neurons, g_i is the i^{th} column of the matrix G, $W_{gi}\in\mathbb{R}^{L_{gi}+1\times n}$, $V_{gi_j}\in\mathbb{R}^{n\times L_{gi}}$ are unknown ideal weight matrices of the MLNN having L_{gi} hidden layer neurons, $i=1...m$, $j=1,2..,N$, $\sigma_f:\mathbb{R}^{Nn}\to\mathbb{R}^{L_f+1}$ and $\sigma_{gi}:\mathbb{R}^{Nn}\to\mathbb{R}^{L_{gi}+1}$ defined as $\sigma_f\triangleq\sigma_f(\sum_{j=1}^{N}V_{f_j}^T x_j)$, $\sigma_{gi}\triangleq\sigma_{gi}(\sum_{j=1}^{N}V_{gi_j}^T x_j)$ are activation functions (sigmoid, hyperbolic tangent etc.), and $\varepsilon_f,\varepsilon_{gi}:\mathbb{R}^{Nn}\to\mathbb{R}^n, i=1...m$ are the function reconstruction errors. Using (2) and Assumption 2, the system in (1) can be represented as

$$\begin{aligned}\dot{x}_1 &= x_2,\\ &\vdots\\ \dot{x}_{N-1} &= x_N,\\ \dot{x}_N &= W_f^T\sigma_f+\varepsilon_f+d+\sum_{i=1}^{m}\left[W_{gi}^T\sigma_{gi}+\varepsilon_{gi}\right]u_i\end{aligned} \tag{3}$$

where $u_i\in\mathbb{R}$ is the i^{th} element of the control input vector u. The following assumptions will be used in the observer development and stability analysis.

Assumption 4. The ideal NN weights are bounded by known positive constants [20], i.e. $\|W_f\|\le\bar{W}_f$, $\left\|V_{f_j}\right\|\le\bar{V}_{f_j}$, $\|W_{gi}\|\le\bar{W}_{gi}$, and $\left\|V_{gi_j}\right\|\le\bar{V}_{gi_j}$, $i=1...m$, $j=1,2..,N$, where $\|\cdot\|$ denotes the Frobenius norm for a matrix and Euclidean norm for a vector.

Assumption 5. The activation functions σ_f,σ_{gi} and their partial derivatives, σ_f',σ_{gi}', σ_f'',σ_{gi}'', $i=1...m$, are bounded [20].

Assumption 6. The function reconstruction errors $\varepsilon_f,\varepsilon_{gi}$, and the respective first partial derivatives are bounded, with $i=1...m$ [20].

The following multi-layer dynamic neural network (MLDNN) architecture is proposed to observe the system in (1)

$$\begin{aligned}\dot{\hat{x}}_1 &= \hat{x}_2,\\ &\vdots\\ \dot{\hat{x}}_{N-1} &= \hat{x}_N,\\ \dot{\hat{x}}_N &= \hat{W}_f^T\hat{\sigma}_f + \sum_{i=1}^{m}\hat{W}_{gi}^T\hat{\sigma}_{gi}u_i + v,\end{aligned} \tag{4}$$

where $\hat{x}=[\hat{x}_1\ \hat{x}_2\ \ldots\hat{x}_N^T]^T \in \mathbb{R}^{Nn}$ is the state of the DNN observer, $\hat{W}_f \in \mathbb{R}^{L_f+1\times n}$, $\hat{V}_{f_j} \in \mathbb{R}^{n\times L_f}, \hat{W}_{gi} \in \mathbb{R}^{L_{gi}+1\times n}$, $\hat{V}_{gi_j} \in \mathbb{R}^{n\times L_{gi}}$, $i = 1...m$, $j = 1, 2.., N$ are the weight estimates, $\hat{\sigma}_f \in \mathbb{R}^{L_f+1}$, and $\hat{\sigma}_{gi} \in \mathbb{R}^{L_{gi}+1}$ are defined as $\hat{\sigma}_f \triangleq \sigma_f(\sum_{j=1}^{N}\hat{V}_{f_j}^T\hat{x}_j)$, $\hat{\sigma}_{gi} \triangleq \sigma_{gi}(\sum_{j=1}^{N}\hat{V}_{gi_j}^T\hat{x}_j)$, and $v \in \mathbb{R}^n$ is a function that is subsequently designed to provide robustness to function reconstruction errors and external disturbances.

Remark 1. In (4), the feedforward NN terms $\hat{W}_f\hat{\sigma}_f$, $\hat{W}_{gi}\hat{\sigma}_{gi}$ use internal feedback of the observer states $\hat{x}$: a DNN structure. The recurrent feedback loop in a DNN enables it to approximate dynamic systems with any arbitrary degree of accuracy [21], [22]. This property motivates the DNN-based observer design. The DNN is automatically trained to estimate the system dynamics by the weight update laws based on the state, weight estimates, and the filter output.

The objective is to prove that the estimated state $\hat{x}$ converges to the system state x. To facilitate the subsequent analysis, the estimation error $\tilde{x}_1 \in \mathbb{R}^n$ is defined as

$$\tilde{x}_1 \triangleq x_1 - \hat{x}_1. \tag{5}$$

To facilitate the subsequent stability analysis and compensate for the lack of direct measurements of x_N, the following filtered estimation errors are defined as

$$\begin{aligned}\tilde{x}_2 &\triangleq \dot{\tilde{x}}_1 + \alpha_1\tilde{x}_1,\\ \tilde{x}_j &\triangleq \dot{\tilde{x}}_{j-1} + \alpha_{j-1}\tilde{x}_{j-1} + \tilde{x}_{j-2},\ \ j = 3, \ldots, N-1\\ r &\triangleq \dot{\tilde{x}}_{N-1} + \alpha\tilde{x}_{N-1} + \eta,\end{aligned} \tag{6}$$

where $\alpha, \alpha_1, \ldots, \alpha_{N-2} \in \mathbb{R}$ are positive constant control gains, and $\eta \in \mathbb{R}^n$ is an output of the dynamic filter [11]

$$\begin{aligned}\eta &= p - (k+\alpha)\tilde{x}_{N-1},\\ \dot{p} &= -(k+2\alpha)p - \tilde{x}_f + ((k+\alpha)^2+1)\tilde{x}_{N-1},\\ \dot{\tilde{x}}_f &= p - \alpha\tilde{x}_f - (k+\alpha)\tilde{x}_{N-1},\\ p(0) &= (k+\alpha)\tilde{x}_{N-1}(0),\quad \tilde{x}_f(0) = 0,\end{aligned} \tag{7}$$

where $\tilde{x}_f \in \mathbb{R}^n$ is an auxiliary output of the filter, $p \in \mathbb{R}^n$ is used as an internal filter variable, and $k \in \mathbb{R}$ is a positive constant gain. The filtered estimation error r is not measurable, since the expression in (6) depends on $\dot{x}$.

Remark 2. The second order dynamic filter to estimate the system velocity was first proposed for the output feedback controller in [11]. The filter in (7) admits the filtered estimation error $\tilde{x}_{N-1}$ as its input and produces two signal outputs $\tilde{x}_f$ and η. The auxiliary signal p is utilized to only generate the signal η without involving the derivative of the estimation error $\dot{\tilde{x}}_{N-1}$ which is unmeasurable. Hence, the filter can be physically implemented. A difficulty to obtain asymptotic estimation is that the filtered estimation error r is not available for feedback. A relationship between the two filter outputs is $\eta = \dot{\tilde{x}}_f + \alpha\tilde{x}_f$, and this relationship is utilized to generate the feedback of r. By taking the time derivative of r, the term $\ddot{x}_f$ appears implicitly inside $\dot{\eta}$. Consequently, the unmeasurable term $\dot{\tilde{x}}_N$ is introduced in a way that it can be replaced by r.

Remark 3. Several observer designs (cf. [7], [23], [24]) exploit a strictly positive real (SPR) condition to show convergence, typically exploiting the Meyer-Kalman-Yakubovich lemma. Designs based on the SPR condition require the dynamic order of the observer to be large. The developed observer avoids this condition by using a dynamic filter to inject the negative feedback of $r(t)$, yielding convergence of the observer.

Taking the derivative of η and using (6) and (7) yields

$$\dot{\eta} = -(k+\alpha)r - \alpha\eta + \tilde{x}_{N-1} - \tilde{x}_f. \tag{8}$$

The closed-loop dynamics of the derivative of the filtered estimation error r in (6) is determined from (3)-(6) and (8) as

$$\begin{aligned}\dot{r} = {} & W_f^T\sigma_f - \hat{W}_f^T\hat{\sigma}_f + \sum_{i=1}^{m}[W_{gi}^T\sigma_{gi} - \hat{W}_{gi}^T\hat{\sigma}_{gi}]u_i + \varepsilon_f\\ & + \sum_{i=1}^{m}\varepsilon_{gi}u_i + d - v + \alpha(r - \alpha\tilde{x}_{N-1} - \eta)\\ & - (k+\alpha)r - \alpha\eta + \tilde{x}_{N-1} - \tilde{x}_f.\end{aligned} \tag{9}$$

Based on the subsequent analysis, the robust disturbance rejection term v is designed to inject cross terms to account for related terms in the stability analysis and a sliding mode term to deal with the disturbances in the system as

$$\begin{aligned}v = {} & -[\gamma(k+\alpha) + 2\alpha]\eta + (\gamma - \alpha^2)\tilde{x}_{N-1}\\ & + \beta_1 sgn(\tilde{x}_{N-1} + \tilde{x}_f),\end{aligned} \tag{10}$$

where $\gamma, \beta_1 \in \mathbb{R}$ are positive constant control gains. By adding and subtracting $W_f^T\sigma_f(\sum_{j=1}^{N}\hat{V}_{f_j}^T x_j) + \hat{W}_f^T\sigma_f(\sum_{j=1}^{N}\hat{V}_{f_j}^T x_j) + \sum_{i=1}^{m}[W_{gi}^T\sigma_{gi}(\sum_{j=1}^{N}\hat{V}_{gi_j}^T x_j) + \hat{W}_{gi}^T\sigma_{gi}(\sum_{j=1}^{N}\hat{V}_{gi_j}^T x_j)]u_i$ and substituting v from (10), the expression in (9) can be rewritten as

$$\begin{aligned}\dot{r} = {} & \tilde{N} + N - kr - \beta_1 sgn(\tilde{x}_{N-1} + \tilde{x}_f)\\ & + \gamma(k+\alpha)\eta - \gamma\tilde{x}_{N-1},\end{aligned} \tag{11}$$

where the auxiliary function $\tilde{N} \in \mathbb{R}^n$ is defined as

$$\begin{aligned}\tilde{N} \triangleq {} & \hat{W}_f^T[\sigma_f(\sum_{j=1}^{N}\hat{V}_{f_j}^T x_j) - \hat{\sigma}_f] + \tilde{x}_{N-1} - \tilde{x}_f\\ & + \sum_{i=1}^{m}\hat{W}_{gi}^T[\sigma_{gi}(\sum_{j=1}^{N}\hat{V}_{gi_j}^T x_j) - \hat{\sigma}_{gi}]u_i,\end{aligned} \tag{12}$$

and $N \in \mathbb{R}^n$ is segregated into two parts as

$$N \triangleq N_1 + N_2. \tag{13}$$

In (13), $N_1 \in \mathbb{R}^n$, and $N_2 \in \mathbb{R}^n$ are defined as

$$N_1 \triangleq \tilde{W}_f^T \sigma'_f [\sum_{j=1}^{N} \tilde{V}_{f_j}^T x_j] + W_f^T O(\sum_{j=1}^{N} \tilde{V}_{f_j}^T x_j)^2 + \sum_{i=1}^{m} \tilde{W}_{gi}^T \sigma'_{gi} [\sum_{j=1}^{N} \tilde{V}_{gi_j}^T x_j] u_i + \varepsilon_f + d + \sum_{i=1}^{m} W_{gi}^T O(\sum_{j=1}^{N} \tilde{V}_{gi_j}^T x_j)^2 u_i + \sum_{i=1}^{m} \varepsilon_{gi} u_i,$$

$$N_2 \triangleq \tilde{W}_f^T \sigma_f (\sum_{j=1}^{N} \hat{V}_{f_j}^T x_j) + \hat{W}_f^T \sigma'_f [\sum_{j=1}^{N} \tilde{V}_{f_j}^T x_j] + \sum_{i=1}^{m} \tilde{W}_{gi}^T \sigma_{gi} (\sum_{j=1}^{N} \hat{V}_{gi_j}^T x_j) u_i + \sum_{i=1}^{m} \hat{W}_{gi}^T \sigma'_{gi} [\sum_{j=1}^{N} \tilde{V}_{gi_j}^T x_j] u_i, \quad (14)$$

where $\tilde{W}_f \triangleq W_f - \hat{W}_f \in \mathbb{R}^{L_f+1 \times n}$, $\tilde{V}_{f_j} \triangleq V_{f_j} - \hat{V}_{f_j} \in \mathbb{R}^{n \times L_f}$, $\tilde{W}_{gi} \triangleq W_{gi} - \hat{W}_{gi} \in \mathbb{R}^{L_{gi}+1 \times n}$, $\tilde{V}_{gi_j} \triangleq V_{gi_j} - \hat{V}_{gi_j} \in \mathbb{R}^{n \times L_{gi}}$, $i = 1...m$, $j = 1, 2.., N$ are the estimate mismatches for the ideal NN weights; $O(\sum_{j=1}^{N} \tilde{V}_{f_j}^T x_j)^2 \in \mathbb{R}^{L_f+1}$, $O(\sum_{j=1}^{N} \tilde{V}_{gi_j}^T x_j)^2 \in \mathbb{R}^{L_{gi}+1}$ are the higher order terms in the Taylor series of the vector functions σ_f, σ_{gi} in the neighborhood of $\sum_{j=1}^{N} \hat{V}_{f_j}^T x_j$ and $\sum_{j=1}^{N} \hat{V}_{gi_j}^T x_j$, respectively, as

$$\sigma_f = \sigma_f(\sum_{j=1}^{N} \hat{V}_{f_j}^T x_j) + \sigma'_f [\sum_{j=1}^{N} \tilde{V}_{f_j}^T x_j] + O(\sum_{j=1}^{N} \tilde{V}_{f_j}^T x_j)^2,$$
$$\sigma_{gi} = \sigma_{gi}(\sum_{j=1}^{N} \hat{V}_{gi_j}^T x_j) + \sigma'_{gi} [\sum_{j=1}^{N} \tilde{V}_{gi_j}^T x_j] + O(\sum_{j=1}^{N} \tilde{V}_{gi_j}^T x_j)^2, \quad (15)$$

where the terms σ'_f, σ'_{gi} are defined as $\sigma'_f \triangleq \sigma'_f(\sum_{j=1}^{N} \hat{V}_{f_j}^T x_j) = d\sigma_f(\theta)/d\theta|_{\theta=\sum_{j=1}^{N} \hat{V}_{f_j}^T x_j}$ and $\sigma'_{gi} \triangleq \sigma'_{gi}(\sum_{j=1}^{N} \hat{V}_{gi_j}^T x_j) = d\sigma_{gi}(\theta)/d\theta|_{\theta=\sum_{j=1}^{N} \hat{V}_{gi_j}^T x_j}$. To facilitate the subsequent analysis, an auxiliary function $\hat{N}_2(\hat{x}_j, \hat{W}_f, \hat{V}_{f_j}, \hat{W}_{gi}, \hat{V}_{gi_j}, t) \in \mathbb{R}^n$ is defined by replacing the terms x_j in N_2 by $\hat{x}_j$, $j = 1, 2.., N$, respectively.

The weight update laws for the DNN in (4) are developed based on the subsequent stability analysis as

$$\dot{\hat{W}}_f = proj[\Gamma_{wf} \hat{\sigma}_f (\tilde{x}_{N-1} + \tilde{x}_f)^T],$$
$$\dot{\hat{V}}_{f_j} = proj[\Gamma_{vf_j} \hat{x}_j (\tilde{x}_{N-1} + \tilde{x}_f)^T \hat{W}_f^T \hat{\sigma}'_f], \quad j = 1..N$$
$$\dot{\hat{W}}_{gi} = proj[\Gamma_{wgi} \hat{\sigma}_{gi} u_i (\tilde{x}_{N-1} + \tilde{x}_f)^T], \quad i = 1...m$$
$$\dot{\hat{V}}_{gi_j} = proj[\Gamma_{vgi_j} \hat{x}_j u_i (\tilde{x}_{N-1} + \tilde{x}_f)^T \hat{W}_{gi}^T \hat{\sigma}'_{gi}], \; i = 1...m, \quad (16)$$

where $\Gamma_{wf} \in \mathbb{R}^{(L_f+1) \times (L_f+1)}$, $\Gamma_{wgi} \in \mathbb{R}^{(L_{gi}+1) \times (L_{gi}+1)}$, $\Gamma_{vf_j}, \Gamma_{vgi_j} \in \mathbb{R}^{n \times n}$, are constant symmetric positive-definite adaptation gains, the terms $\hat{\sigma}'_f, \hat{\sigma}'_{gi}$ are defined as $\hat{\sigma}'_f \triangleq \sigma'_f(\sum_{j=1}^{N} \hat{V}_{f_j}^T \hat{x}_j) = d\sigma_f(\theta)/d\theta|_{\theta=\sum_{j=1}^{N} \hat{V}_{f_j}^T \hat{x}_j}$, $\hat{\sigma}'_{gi} \triangleq \sigma'_{gi}(\sum_{j=1}^{N} \hat{V}_{gi_j}^T \hat{x}_j) = d\sigma_{gi}(\theta)/d\theta|_{\theta=\sum_{j=1}^{N} \hat{V}_{gi_j}^T \hat{x}_j}$, and $proj(\cdot)$ is a smooth projection operator [25], [26] used to guarantee that the weight estimates $\hat{W}_f, \hat{V}_{f_j}, \hat{W}_{gi}$, and $\hat{V}_{gi_j}$ remain bounded.

Using (5)-(7), Assumptions 1-5, the $proj(\cdot)$ algorithm in (16) and the Mean Value Theorem, the auxiliary function $\tilde{N}$ in (12) can be upper-bounded as

$$\left\| \tilde{N} \right\| \leq \zeta_1 \left\| z \right\|, \quad (17)$$

where $z(\tilde{x}_1, \ldots, \tilde{x}_{N-1}, \tilde{x}_f, \eta, r) \in \mathbb{R}^{(N+2)n}$ is defined as

$$z \triangleq [\tilde{x}_1^T \ldots \tilde{x}_{N-1}^T \; \tilde{x}_f^T \; \eta^T \; r^T]^T. \quad (18)$$

Based on (5)-(7), Assumptions 1-6, the Taylor series expansion in (15) and the weight update laws in (16), the following bounds can be developed

$$\|N_1\| \leq \zeta_2, \;\; \|N_2\| \leq \zeta_3, \;\; \left\| \tilde{N}_2 \right\| \leq \zeta_5 \left\| z \right\|,$$
$$\left\| \dot{N} \right\| \leq \zeta_4 + \rho(\|z\|) \left\| z \right\|, \quad (19)$$

where $\zeta_i \in \mathbb{R}, i = 1...5$, are computable positive constants, $\rho \in \mathbb{R}$ is a positive, globally invertible, non-decreasing function, and $\tilde{N}_2 \triangleq N_2 - \hat{N}_2$.

To facilitate the subsequent stability analysis, let $y \in \mathbb{R}^{(N+2)n+2}$ be defined as

$$y \triangleq [z^T \quad \sqrt{P} \quad \sqrt{Q}]^T, \quad (20)$$

and let $\mathcal{D} \subset \mathbb{R}^{(N+2)n+2}$ be the open and connected set

$$\mathcal{D} \triangleq \left\{ y \in R^{(N+2)n+2} | \; \|y\| < \rho^{-1}(\lambda - \frac{\zeta_1^2}{4\sqrt{2}k_2}) \right\}.$$

In (20), the auxiliary function $P \in \mathbb{R}$ is a generalized Filippov solution to the differential equation

$$\dot{P} \triangleq -r^T(N_1 - \beta_1 sgn(\tilde{x}_{N-1} + \tilde{x}_f)) - (\dot{\tilde{x}}_{N-1} + \dot{\tilde{x}}_f)^T N_2, + \sqrt{2}\rho(\|z\|) \left\| z \right\|^2,$$
$$P(0) \triangleq \beta_1 \sum_{i=1}^{n} \left| \tilde{x}_{N-1_i}(0) + \tilde{x}_{f_i}(0) \right| - (\tilde{x}_{N-1}(0) + \tilde{x}_f(0))^T N(0), \quad (21)$$

where the subscript $i = 1, 2, .., n$ denotes the i^{th} element of $\tilde{x}_{N-1}(0)$ or $\tilde{x}_f(0)$, and $\beta_1 \in \mathbb{R}$ is a positive constant chosen according to the sufficient condition

$$\beta_1 > \max(\zeta_2 + \zeta_3, \zeta_2 + \frac{\zeta_4}{\alpha}), \quad (22)$$

where $\zeta_i, i = 2, 3, 4$ are introduced in (19). Provided the sufficient condition in (22) is satisfied, $P \geq 0$ (see [27]). The auxiliary function $Q \in \mathbb{R}$ in (20) is defined as

$$Q \triangleq \frac{\alpha}{2} \left[\sum_{i=1}^{m} tr(\tilde{W}_{gi}^T \Gamma_{wg_i}^{-1} \tilde{W}_{gi}) + \sum_{j=1}^{N} tr(\tilde{V}_{f_j}^T \Gamma_{vf_j}^{-1} \tilde{V}_{f_j}) + tr(\tilde{W}_f^T \Gamma_{wf}^{-1} \tilde{W}_f) + \sum_{j=1}^{N} \sum_{i=1}^{m} tr(\tilde{V}_{gi_j}^T \Gamma_{vgi_j}^{-1} \tilde{V}_{gi_j}) \right] \quad (23)$$

where $tr(\cdot)$ denotes the trace of a matrix. Since the gains $\Gamma_{wf}, \Gamma_{wgi}, \Gamma_{vf_j}, \Gamma_{vgi_j}$ are symmetric, positive-definite matrices, $Q \geq 0$.

III. Lyapunov Stability Analysis For DNN-based Observer

Theorem 1. *The DNN-based observer proposed in (4) along with its weight update laws in (16) ensures semi-global asymptotic estimation in sense that*

$$\|\tilde{x}_j(t)\| \to 0 \text{ and } \|x_N(t) - \hat{x}_N(t)\| \to 0 \text{ as } t \to \infty,$$

with $j = 1 \ldots (N-1)$, provided the control gain $k = k_1 + k_2$ introduced in (7) is selected sufficiently large based on the initial conditions of the states[1]*, the gain condition in (22) is satisfied, and the following sufficient conditions are satisfied*

$$\alpha_j, \alpha,\ k_1 > \frac{1}{2},\ \gamma > \frac{2\alpha^2\zeta_5^2 + 1}{2\alpha - 1},\ \text{and } \lambda > \frac{\zeta_1^2}{4\sqrt{2}k_2} \quad (24)$$

where

$$\lambda \triangleq \frac{1}{\sqrt{2}}\left[\min(\alpha\gamma - \frac{\gamma}{2} - \alpha^2\zeta_5^2, \gamma(\alpha_j - \frac{1}{2}), k_1) - \frac{1}{2}\right], \quad (25)$$

and ζ_1, ζ_5 are introduced in (17) and (19), respectively.

Proof: Consider the Lyapunov candidate function $V_L : \mathcal{D} \to \mathbb{R}$, which is a Lipschitz continuous positive definite function defined as

$$V_L \triangleq \frac{\gamma}{2}\sum_{j=1}^{N-1}\tilde{x}_j^T\tilde{x}_j + \frac{\gamma}{2}\tilde{x}_f^T\tilde{x}_f + \frac{\gamma}{2}\eta^T\eta + \frac{1}{2}r^Tr + P + Q, \quad (26)$$

which satisfies the following inequalities:

$$U_1(y) \leq V_L(y) \leq U_2(y), \quad (27)$$

where $U_1, U_2 : \mathbb{R}^{(N+2)n+2} \to \mathbb{R}$ are continuous positive definite functions defined as $U_1(y) \triangleq \min(\frac{\gamma}{2}, \frac{1}{2})\|y\|^2$, $U_2(y) \triangleq \max(\frac{\gamma}{2}, 1)\|y\|^2$.

Let $\dot{y} = h(y,t)$ represent the closed-loop differential equations in (6)-(8), (11), (16) and (21), where $h(y,t) \in \mathbb{R}^{(N+2)n+2}$ denotes the right-hand side of the closed-loop error signals. Using Filippov's theory of differential inclusions [28]–[31], the existence of solutions can be established for $\dot{y} \in K[h](y,t)$, where $K[h] \triangleq \bigcap_{\delta>0}\bigcap_{\mu M=0}\overline{co}h(B(y,\delta) \setminus M, t)$, where $\bigcap_{\mu M=0}$ denotes the intersection of all sets M of Lebesgue measure zero, $\overline{co}$ denotes convex closure, and $B(y,\delta) = \left\{w \in R^{(N+2)n+2} | \ \|y - w\| < \delta\right\}$. The generalized time derivative of (26) exists almost everywhere (a.e.), i.e. for almost all $t \in [t_0, t_f]$, and $\dot{V}_L(y(t)) \in^{a.e.} \dot{\tilde{V}}_L(y(t))$, where $\dot{\tilde{V}}_L = \bigcap_{\xi \in \partial V_L(y)} \xi^T K[\Psi]^T$, ∂V_L is the generalized gradient of V_L [32], and $\Psi \triangleq \left[\dot{\tilde{x}}_1^T \ldots \dot{\tilde{x}}_{N-1}^T\ \dot{\tilde{x}}_f^T\ \dot{\eta}^T\ \dot{r}^T\ \frac{1}{2}P^{-\frac{1}{2}}\dot{P}\ \frac{1}{2}Q^{-\frac{1}{2}}\dot{Q}\right]$. Since V_L is continuously differentiable, $\dot{\tilde{V}}_L$ can be simplified as [33] $\dot{\tilde{V}}_L = \nabla V_L^T K[\Psi]^T = \left[\gamma\tilde{x}_1^T \ldots \gamma\tilde{x}_{N-1}^T\ \gamma\tilde{x}_f^T\ \gamma\eta^T\ r^T\ 2P^{\frac{1}{2}}\ 2Q^{\frac{1}{2}}\right]K[\Psi]^T$. Using the calculus for $K[\cdot]$ from [34] (Theorem 1, Properties $2, 5, 7$), and substituting the dynamics from (6)-(8), (11), (21), and (23) and adding and subtracting $\alpha(\tilde{x}_{N-1} + \tilde{x}_f)^T\hat{N}_2$ and using (14), $\dot{\tilde{V}}_L$ can be rewritten as

$$\begin{aligned}
\dot{\tilde{V}}_L \subset\ & \gamma\tilde{x}_1^T(\tilde{x}_2 - \alpha_1\tilde{x}_1) + \gamma\tilde{x}_{N-1}^T(r - \alpha\tilde{x}_{N-1} - \eta) \\
& + \gamma\sum_{j=2}^{N-2}\tilde{x}_j^T(\tilde{x}_{j+1} - \alpha_j\tilde{x}_j - \tilde{x}_{j-1}) + \gamma\tilde{x}_f^T(\eta - \alpha\tilde{x}_f) \\
& + \gamma\eta^T[-(k+\alpha)r - \alpha\eta + \tilde{x}_{N-1} - \tilde{x}_f] \\
& - kr^Tr - \gamma r^T\tilde{x}_{N-1} - \alpha(\tilde{x}_{N-1} + \tilde{x}_f)^T\hat{N}_2 + r^T\tilde{N} \\
& + r^T[N - \beta_1 K[sgn(\tilde{x}_{N-1} + \tilde{x}_f)] + \gamma(k+\alpha)\eta] \\
& - r^T(N_1 - \beta_1 K[sgn(\tilde{x}_{N-1} + \tilde{x}_f)]) \\
& + \alpha(\tilde{x}_{N-1} + \tilde{x}_f)^T\left\{\tilde{W}_f^T\hat{\sigma}_f + \hat{W}_f^T\hat{\sigma}'_f[\sum_{j=1}^{N}\tilde{V}_{f_j}^T\hat{x}_j]\right. \\
& \left. + \sum_{i=1}^{m}\tilde{W}_{gi}^T\hat{\sigma}_{gi}u_i + \sum_{i=1}^{m}\hat{W}_{gi}^T\hat{\sigma}'_{gi}[\sum_{j=1}^{N}\tilde{V}_{gi_j}^T\hat{x}_j]u_i\right\} \\
& - \alpha\left[\sum_{i=1}^{m}tr(\tilde{W}_{gi}^T\Gamma_{wg}^{-1}\dot{\hat{W}}_{gi}) + \sum_{j=1}^{N}tr(\tilde{V}_{f_j}^T\Gamma_{vf_j}^{-1}\dot{\hat{V}}_{f_j})\right. \\
& \left. + tr(\tilde{W}_f^T\Gamma_{wf}^{-1}\dot{\hat{W}}_f) + \sum_{i=1}^{m}\sum_{j=1}^{N}tr(\tilde{V}_{gi_j}^T\Gamma_{vgi_j}^{-1}\dot{\hat{V}}_{gi_j})\right] \\
& - (\dot{\tilde{x}}_{N-1} + \dot{\tilde{x}}_f)^TN_2 + \sqrt{2}\rho(\|z\|)\|z\|^2.
\end{aligned} \quad (28)$$

Using the fact that $K[sgn(\tilde{x}_{N-1} + \tilde{x}_f)] = SGN(\tilde{x}_{N-1} + \tilde{x}_f)$ [34], such that $SGN(\tilde{x}_{N-1_i} + \tilde{x}_{f_i}) = 1$ if $(\tilde{x}_{N-1_i} + \tilde{x}_{f_i}) > 0$, $[-1, 1]$ if $(\tilde{x}_{N-1_i} + \tilde{x}_{f_i}) = 0$, and -1 if $(\tilde{x}_{N-1_i} + \tilde{x}_{f_i}) < 0$ (the subscript i denotes the i^{th} element), the set in (28) reduces to the scalar inequality, since the RHS is continuous a.e., i.e., the RHS is continuous except for the Lebesgue measure zero set of times when[2] $r^T SGN(\tilde{x}_{N-1}(t) + \tilde{x}_f(t)) - r^T SGN(\tilde{x}_{N-1}(t) + \tilde{x}_f(t)) = 0$. Substituting the weight update laws in (16) and canceling common terms yields

$$\begin{aligned}
\dot{\tilde{V}}_L \overset{a.e.}{\leq} & -\gamma\sum_{j=1}^{N-2}\alpha_j\tilde{x}_j^T\tilde{x}_j - \gamma\alpha\tilde{x}_{N-1}^T\tilde{x}_{N-1} + \gamma\tilde{x}_{N-2}^T\tilde{x}_{N-1} \\
& - \gamma\alpha\tilde{x}_f^T\tilde{x}_f - \gamma\alpha\eta^T\eta - kr^Tr + r^T\tilde{N} \\
& + \alpha(\tilde{x}_{N-1} + \tilde{x}_f)^T\tilde{N}_2 + \sqrt{2}\rho(\|z\|)\|z\|^2.
\end{aligned} \quad (29)$$

[1] See the subsequent proof

[2] Let $\Phi \triangleq \tilde{x}_{N-1} + \tilde{x}_f$. The set of times $\Lambda \triangleq \left\{t \in [0,\infty) : r(t)^T K[sgn(\Phi(t))] - r(t)^T K[sgn(\Phi(t))] \neq \{0\}\right\} \subset [0,\infty)$ is equal to the set of times $\{t : \Phi(t) = 0 \wedge r(t) \neq 0\}$. Using the fact that $\eta = \dot{\tilde{x}}_f + \alpha\tilde{x}_f$, r can be expressed as $r = \dot{\Phi} + \alpha\Phi$. Thus, the set Λ can also be represented by $\left\{t : \Phi(t) = 0 \wedge \dot{\Phi}(t) \neq 0\right\}$. Since $\phi : [0,\infty) \to \mathbb{R}^n$ is continuously differentiable, it can be shown that the set of time instances $\left\{t : \Phi(t) = 0 \wedge \dot{\Phi}(t) \neq 0\right\}$ is isolated, and thus, measure zero; hence, Λ is measure zero.

Using (17), (19), the facts that

$$\gamma\tilde{x}_{N-2}^T\tilde{x}_{N-1} \leq \frac{\gamma}{2}\left\|\tilde{x}_{N-1}\right\|^2 + \frac{\gamma}{2}\left\|\tilde{x}_{N-2}\right\|^2$$
$$\alpha\zeta_5\left\|\tilde{x}_{N-1} + \tilde{x}_f\right\|\left\|z\right\| \leq \alpha^2\zeta_5^2\left\|\tilde{x}_{N-1}\right\|^2 + \alpha^2\zeta_5^2\left\|\tilde{x}_f\right\|^2 + \frac{1}{2}\left\|z\right\|^2,$$

substituting $k = k_1 + k_2$, and completing the squares, the expression in (29) can be further bounded as

$$\dot{\tilde{V}}_L \overset{a.e.}{\leq} -\gamma\sum_{j=1}^{N-2}\left(\alpha_j - \frac{1}{2}\right)\left\|\tilde{x}_j\right\|^2 - \alpha(\gamma - \alpha\zeta_5^2)\left\|\tilde{x}_f\right\|^2 - (\alpha\gamma - \frac{\gamma}{2} - \alpha^2\zeta_5^2)\left\|\tilde{x}_{N-1}\right\|^2 - \alpha\gamma\left\|\eta\right\|^2 - k_1\left\|r\right\|^2 + \left(\frac{1}{2} + \frac{\zeta_1^2}{4k_2} + \sqrt{2}\rho(\left\|z\right\|)\right)\left\|z\right\|^2.$$

Provided the sufficient conditions in (24) are satisfied, the above expression can be rewritten as

$$\dot{\tilde{V}}_L \overset{a.e.}{\leq} -\sqrt{2}(\lambda - \frac{\zeta_1^2}{4\sqrt{2}k_2} - \rho(\left\|z\right\|))\left\|z\right\|^2 \overset{a.e.}{\leq} -U(y) \quad \forall y \in \mathcal{D}, \tag{30}$$

where λ is defined in (25), and $U(y) = c\left\|z\right\|^2$, for some positive constant c, is a continuous positive semi-definite function which is defined on $\mathcal{D}$. The inequalities in (27) and (30) show that $V_L \in \mathcal{L}_\infty$; hence, $\tilde{x}_j, \tilde{x}_f, \eta, r, P$ and $Q \in \mathcal{L}_\infty$, with $\forall j = 1 \ldots N-1$; (6)-(8) are used to show that $\dot{\tilde{x}}_j, \dot{\tilde{x}}_f, \dot{\eta} \in \mathcal{L}_\infty$, with $\forall j = 1 \ldots N-1$. Since $x_j \in \mathcal{L}_\infty$ by Assumption 1, $\hat{x}_j \in \mathcal{L}_\infty$ using (5). Since $\tilde{x}_j, \tilde{x}_f, \eta \in \mathcal{L}_\infty$, using (10), $v \in \mathcal{L}_\infty$. Since $W_f, W_{gi}, \sigma_f, \sigma_{gi}, \varepsilon_f, \varepsilon_{gi} \in \mathcal{L}_\infty, i = 1...m$, by Assumptions 4-6, the control input u and the disturbance d are bounded by Assumptions 2-3, and $\hat{W}_f, \hat{W}_{gi} \in \mathcal{L}_\infty, i = 1...m$, by the use of the $proj(\cdot)$ algorithm, from (9), $\dot{r} \in \mathcal{L}_\infty$; then $\dot{z} \in \mathcal{L}_\infty$, by using (18). Let $S \subset \mathcal{D}$ denote a set defined as

$$S \triangleq \left\{y \in \mathcal{D} | U_2(y) < \varepsilon_1(\rho^{-1}(\lambda - \frac{\zeta_1^2}{4\sqrt{2}k_2}))^2\right\}. \tag{31}$$

From (30), [35, Corollary 1] can be invoked to show that $c\left\|z\right\|^2 \to 0$ as $t \to \infty$, $\forall y(0) \in \mathcal{S}$. Based on the definition of z the following result can be proven

$$\left\|\tilde{x}_j(t)\right\|, \left\|\eta(t)\right\|, \left\|r(t)\right\| \to 0 \text{ as } t \to \infty \quad \forall y(0) \in \mathcal{S},\ \forall j = 1 \ldots N-1.$$

From (6), it can be further shown that

$$\left\|x_N(t) - \hat{x}_N(t)\right\| \to 0 \text{ as } t \to \infty \quad \forall y(0) \in \mathcal{S}.$$

Note that the region of attraction in (31) can be made arbitrarily large to include any initial condition by increasing the control gains k, γ, α_j and α (i.e., a semi-global type of stability result). ■

IV. Experiment and Simulation Results

Experiments and simulations on a two-link robot manipulator, a second-order system, are performed to compare the proposed method with several other estimation methods. The testbed is composed of a two-link direct drive revolute robot consisting of two aluminum links. The motor encoders provide position measurements with a resolution of 614400 pulses/revolution. The following dynamics of the testbed are considered for the simulations and experiments:

$$M(x)\ddot{x} + V_m(x,\dot{x})\dot{x} + F_d\dot{x} + F_s(\dot{x}) = u, \tag{32}$$

where $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ are the angular positions (rad) and $\dot{x} = \begin{bmatrix} \dot{x}_1 & \dot{x}_2 \end{bmatrix}^T$ are the angular velocities (rad/s) of the two links respectively. In (32), M is the inertia matrix and V_m is the centripetal-Coriolis matrix, defined as

$$M(x) \triangleq \begin{bmatrix} p_1 + 2p_3c_2 & p_2 + p_3c_2 \\ p_2 + p_3c_2 & p_2 \end{bmatrix},$$
$$V_m(x,\dot{x}) \triangleq \begin{bmatrix} -p_3s_2\dot{x}_2 & -p_3s_2(\dot{x}_1 + \dot{x}_2) \\ p_3s_2\dot{x}_1 & 0 \end{bmatrix}. \tag{33}$$

In (32) and (33), parameters for simulation are chosen as the best-guess of the testbed model as $p_1 = 3.473\ kg \cdot m^2$, $p_2 = 0.196\ kg \cdot m^2$, $p_3 = 0.242\ kg \cdot m^2$, $c_2 = cos(x_2)$, $s_2 = sin(x_2)$. $F_d = diag\{5.3, 1.1\}\, Nm \cdot sec$ and $F_s(\dot{x}) = diag\{8.45tanh(\dot{x}_1), 2.35tanh(\dot{x}_2)\}\, Nm$ are the models for dynamic and static friction, respectively. The system in (32) can be rewritten as

$$\ddot{x} = f(x,\dot{x}) + G(x,\dot{x})u + d,$$

where $d \in \mathbb{R}^2$ is the additive exogenous disturbance and $f : \mathbb{R}^4 \to \mathbb{R}^2$, and $G : \mathbb{R}^4 \to \mathbb{R}^4$ are defined as

$$f(x,\dot{x}) = M^{-1}(-V_m - F_d)\dot{x} - F_s,\ G(x,\dot{x}) = M^{-1}.$$

The control input is chosen as a PD controller to track a desired trajectory $x_d(t) = [0.5sin(2t)\ 0.5cos(2t)]^T$, as $u = 20(x - x_d) + 10(\dot{x} - \dot{x}_d)$, where the angular velocity $\dot{x}$ used only in the control law is determined numerically by a standard backwards difference algorithm.

The objective is to design an observer $\dot{\hat{x}}$ to asymptotically estimate the angular velocities $\dot{x}$ using only the measurements of angular positions x. The control gains for the experiment are selected as $k = 7$, $\alpha = 7$, $\gamma = 8$, $\beta_1 = 6$, and $\Gamma_{wf} = \Gamma_{wg1} = \Gamma_{wg2} = 3\mathbb{I}_{8\times 8}$, $\Gamma_{vf} = \Gamma_{vg1} = \Gamma_{vg2} = 3\mathbb{I}_{2\times 2}$, where $\mathbb{I}_{n\times n}$ denotes an identity matrix of appropriate dimensions. The NNs are designed to have seven hidden layer neurons and the NN weights are initialized as uniformly distributed random numbers in the interval $[-1, 1]$. The initial conditions of the system and the identifier are chosen as $x_0 = [0\ 0]^T$, $\dot{x}_0 = [0\ 0]^T$ and $\hat{x}_0 = \dot{\hat{x}}_0 = [0, 0]^T$, respectively.

A global asymptotic velocity observer for uncertain nonlinear systems was developed by Xian et al. [11] as

$$\dot{\hat{x}} = p + K_0\tilde{x},\ \ \dot{p} = K_1sgn(\tilde{x}) + K_2\tilde{x},$$

and a high gain (HG) observer that is asymptotic as the gain goes to infinity was developed in [12] as

$$\dot{\hat{x}} = z_h + \frac{\alpha_{h1}}{\varepsilon_{h1}}\tilde{x},\ \ \dot{z}_h = \frac{\alpha_{h2}}{\varepsilon_{h2}}\tilde{x}.$$

Both these designs are based on a purely robust feedback strategy. One of the contributions of this work is the addition of a feed-forward adaptive component to compensate for the

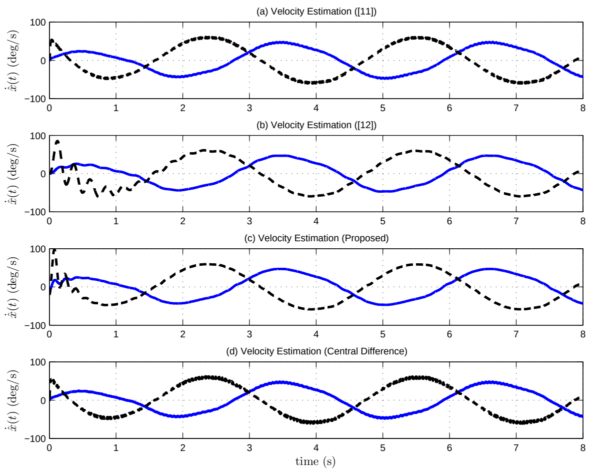

Fig. 1. Velocity estimate $\hat{\dot{x}}$ using (a) [11], (b) [12], (c) the proposed method, and (d) the center difference method on a two-link experiment testbed (solid line: Link 1, dashed line: Link 2).

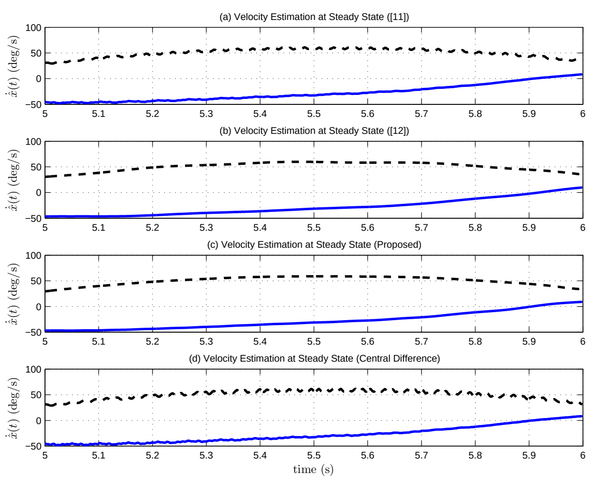

Fig. 2. The steady-state velocity estimate $\hat{\dot{x}}$ using (a) [11], (b) [12], (c) the proposed method, and (d) the center difference method on a two-link experiment testbed (solid line: Link 1, dashed line: Link 2).

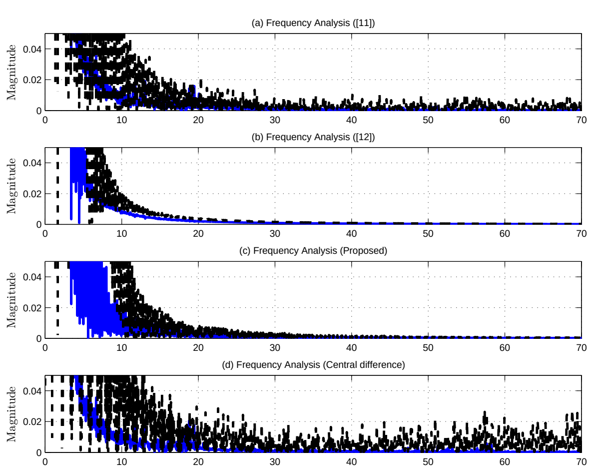

Fig. 3. Frequency analysis of velocity estimation $\hat{\dot{x}}$ using (a) [11], (b) [12], (c) the proposed method, and (d) the center difference method on a two-link experiment testbed (solid line: Link 1, dashed line: Link 2).

uncertain dynamics. To gauge the benefit of this approach, the proposed observer is compared with the observers in [11] and [12]. Control gains for the observer in [11] are chosen as $K_0 = 10$, $K_1 = 6$, and $K_2 = 10$, and control gains for the HG observer are chosen as $\alpha_1 = 0.6$, $\alpha_2 = 25$, $\varepsilon_1 = 0.01$, and $\varepsilon_2 = 0.015$. To make the comparison feasible, the gains of all observers are tuned to get the steady state RMS of position estimation errors to be approximately equal to 0.17 for a settling time of 1 second. The experiment results for the velocity estimators in [11], [12], and the proposed method are compared with the central difference algorithm. The results are shown in Figs. 1 and 2. It is observed that the velocity estimates of the proposed observer and observer in [12] look similar, but the transient response of the proposed method is improved over the observer in [12]; moreover, both methods have lower frequency content than the observer in [11] and the central difference method. To illustrate the lower frequency response of the proposed method compared to [11] and the central difference method, the frequency analysis plots of the experiment results are shown in Fig. 3. Fig. 3 illustrates that the velocity estimation using [11] and central difference methods include higher frequency signals than the proposed method or the approach in [12].

Given the lack of velocity sensors in the two-link experiment testbed to verify the velocity estimates, a simulation was performed using the dynamics in (32). To examine the effect of noise, white Gaussian noise with SNR 60 dB is added to the position measurements. Fig. 4 shows the simulation results for the steady-state velocity estimation errors and the respective frequency analysis for the velocity estimate of the observer in [11], the observer in [12], the developed method, and the central difference method. Table I gives a comparison of the transient and steady state RMS velocity estimation errors for these different methods. Results of the standard numerical central differentiation algorithm are significantly worse than the other methods in the presence of noise as seen from Fig. 4 and Table I. Although, simulation results for [12] and the developed method are comparable, differences exist in the structure of the observers and proof of convergence of the estimates. The observer in [12] is a purely robust feedback technique and the estimation result is proven to be asymptotic as the gains tend to infinity. On the other hand, the proposed method is a robust adaptive observer with a DNN structure to learn the system uncertainties, combining a dynamic filter and a robust sliding mode structure, thus guaranteeing asymptotic convergence with finite gains. Further, the observer in [11] is also a purely robust feedback method, where all uncertainties are damped out by a sliding mode term resulting in higher frequency velocity estimates than the developed observer, as seen from both experiment and simulation results.

V. Conclusion

A novel design of an adaptive observer using DNNs for high-order uncertain nonlinear systems is proposed. The DNN works in conjunction with a dynamic filter without an off-line training phase. A sliding feedback term is added

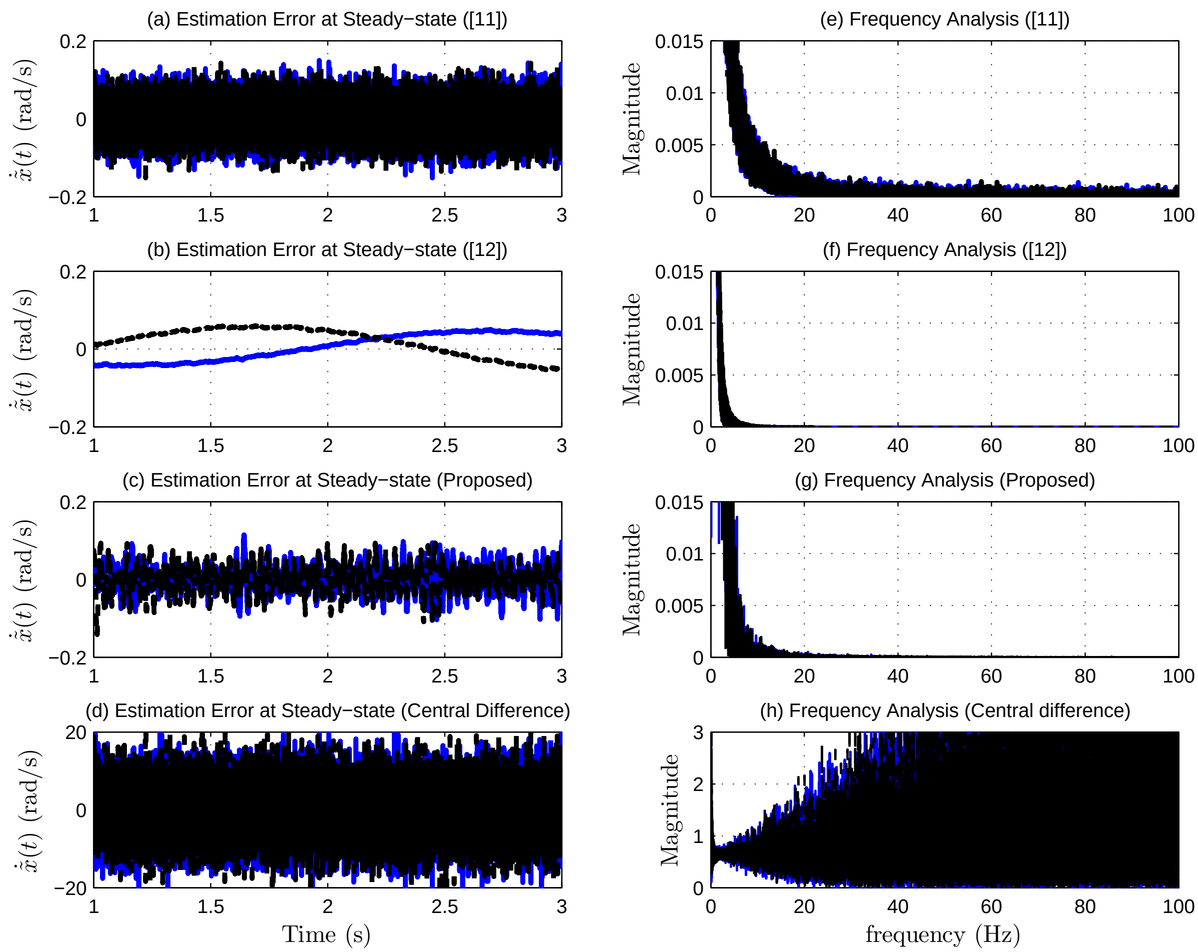

Fig. 4. The steady-state velocity estimation error $\dot{\tilde{x}}$ using (a) [11], (b) [12], (c) the proposed method, and (d) the center difference method on simulations, in presence of sensor noise (SNR 60dB). The right figures (e)-(h) indicate the respective frequency analysis of velocity estimation $\dot{\tilde{x}}$ (solid line: Link 1, dashed line: Link 2).

TABLE I
TRANSIENT ($t = 0 - 1$ SEC) AND STEADY STATE ($t = 1 - 10$ SEC) VELOCITY ESTIMATION ERRORS $\dot{\tilde{x}}$ FOR DIFFERENT VELOCITY ESTIMATION METHODS IN PRESENCE OF NOISE 50DB.

	Central difference	Method in [11]	Method in [12]	Proposed
Transient RMS Error	66.2682	0.1780	0.1040	0.1309
Steady State RMS Error	8.1608	0.0565	0.0538	0.0504

to the DNN structure to account for reconstruction errors and external disturbances. The observation states are proven to asymptotically converge to the system states. Simulations and experiments on a two-link robot manipulator show the improvement of the proposed method in comparison to several other estimation methods.

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