Lyapunov-Based Control of an Uncertain Euler-Lagrange System with Time-Varying Input Delay

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***Abstract*—This paper examines control of a general class of uncertain nonlinear Euler-Lagrange systems with time-varying input delay and additive bounded disturbances. A Lyapunov-based stability analysis utilizing Lyapunov-Krasovskii functionals is provided to prove semi-global uniformly ultimately bounded tracking. Simulation results demonstrate the robustness of the control design with respect to the delay.**

I. Introduction

Time-delays are prevalent in nature and many engineering systems and have been well documented in literature (cf. [1]–[4] and relatively recent monographs such as [5]–[10]). Delays in the control input can be found in many well-known and documented applications such as digital implementation of a continuous control signal, internal combustion engines, computer controlled financial markets, chemical process control, automotive systems, traffic management and teleoperated robotic systems, as well as biological processes such as force production in muscle and control of cardiovascular system.

Few mathematical tools exist that can be used to develop predictive controllers that compensate for input delays. Of these tools (namely Smith predictors [11], Artstein model reduction [12], and finite spectrum assignment [13]), few variations have been developed that can compensate for uncertain nonlinear systems. Methods which solve the input delay problem for uncertain nonlinear systems with known and unknown constant time-delays have been studied in [14]–[20]. However, due to uncertainties in the inherent nature of real world systems, it is often more practical to consider time-varying or state-dependent time-delays in the control.

Control methods for time-varying input-delayed systems with linear plant models have been studied extensively. Discrete predictor-based techniques have been developed for linear systems with time-varying input delay in [21], where small bounded uncertainties in the system parameters, delay, and sampling instants are considered. A delayed feedback controller was developed in [22] for uncertain linear systems with a time-varying input delay based on a reduction method. A robust control method for uncertain linear systems with time-varying input delays was developed in [23], which combines a novel Lyapunov-Krasovskii (LK) functional and a neutral transformation to obtain sufficient conditions for closed-loop robustness. Predictive controllers have also been developed under the assumption that input delay systems can be represented by hyperbolic partial differential equations (cf. [3], [6] and references therein). This fact is exploited in [24] to design controllers for actuator delayed linear systems where time delayed system is modeled as an ordinary differential equation (ODE) - partial differential equation (PDE) cascade using an infinite dimensional transformation where the non-delayed input acts at the PDE boundary.

Linearized controllers have been developed for nonlinear systems (cf. [25], [26]), but because the stability of the closed loop system is only valid within a region around the point of linearization, a complete nonlinear control solution to the time-varying input delay problem is still motivated.

A finite-time stabilizing controller developed in [27] compensates for time-varying input delays in nonlinear systems with triangular structures using an integrator backstepping technique. More recently, Bekiaris-Liberis and Krstic [28] extended the results in [24] and [18] to develop a control method for forward complete nonlinear systems with time-varying input delays. Under the assumption that the plant is asymptotically stable in the absence of the input delay, an invertible infinite dimensional backstepping transformation is used to yield an asymptotically stable system. While these results have been successful for certain classes of nonlinear input-delayed systems, the applicability of the methods to general uncertain Euler-Lagrange dynamics is not clear. Motivated by this issue, [29] provided a transformation to convert an Euler-Lagrange system into a forward-complete system, but such a transformation requires exact model knowledge of the Euler-Lagrange dynamics; thus, the technique is not applicable when the system parameters are unknown or the dynamics are uncertain. This implies that methods developed for forward-complete systems with input delays may not be applicable to uncertain Euler-Lagrange systems. Based on these findings, our previous work in [19] developed a continuous controller for uncertain Euler-Lagrange systems with constant input delays capable of achieving semi-global uniformly ultimately bounded tracking.

In this paper, a control method is developed to compensate for time-varying input delays in uncertain nonlinear Euler-Lagrange systems with additive disturbances, under

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the assumption that the time-delay is bounded and slowly varying. As in our previous work, LK functionals are used to facilitate the design and analysis of a control method that can compensate for the input delay. Since the LK functionals contain time-varying delay terms, additional complexities are introduced into the analysis. Techniques used to compensate for the time-varying delay result in new sufficient control conditions that depend on the length of the delay as well as the rate of delay. The developed controller achieves semi-global uniformly ultimately bounded tracking despite the time-varying input delay, parametric uncertainties and additive bounded disturbances in the plant dynamics. A numerical simulation for a two-link robot manipulator with time-varying input delay is provided to examine performance of the developed controller.

II. Dynamic Model and Properties

Consider the following input-delayed Euler-Lagrange dynamics

$$M(q)\,\ddot{q}+V_m(q,\dot{q})\,\dot{q}+G(q)+F(\dot{q})+d\,(t)=u\,(t-\tau\,(t)) \quad (1)$$

where $M\,(q)\in\mathbb{R}^{n\times n}$ denotes a generalized inertia matrix, $V_m\,(q,\dot{q})\in\mathbb{R}^{n\times n}$ denotes a generalized centripetal-Coriolis matrix, $G\,(q)\in\mathbb{R}^n$ denotes a generalized gravity vector, $F\,(\dot{q})\in\mathbb{R}^n$ denotes generalized friction, $d\,(t)\in\mathbb{R}^n$ denotes an exogenous disturbance, $u\,(t-\tau\,(t))\in\mathbb{R}^n$ represents the generalized delayed input control vector, where $\tau\,(t)\in\mathbb{R}$ is a non-negative time-varying delay, and $q\,(t)\,,\dot{q}\,(t)\,,\ddot{q}\,(t)\in\mathbb{R}^n$ denote the generalized states.

The subsequent development is based on the assumption that $q\,(t)\,,\dot{q}\,(t)$ are measurable outputs, $M\,(q)$, $V_m\,(q,\dot{q})$, $G\,(q)$, $F\,(\dot{q})$, $d\,(t)$ are unknown, the time-varying input delay is known and the control input vector and its past values (i.e., $u\,(t-\theta)\,\forall\theta\in[0\ \tau\,(t)]$) are measurable. Throughout the paper, a time dependent delayed function is denoted as $\zeta\,(t-\tau\,(t))$ or ζ_τ. Additionally, the following assumptions and properties will be exploited.

Property 1. The inertia matrix $M\,(q)$ is symmetric positive-definite, and satisfies the following inequality:

$$\underline{m}\,\|\xi\|^2\leq\xi^TM\xi\leq\bar{m}\,\|\xi\|^2\,,\quad\forall\xi\in\mathbb{R}^n$$

where $\underline{m},\bar{m}\in\mathbb{R}^+$ are known constants and $\|\cdot\|$ denotes the standard Euclidean norm.

Assumption 1. The nonlinear disturbance term and its first time derivative (i.e., $d\,(t)\,,\dot{d}\,(t)$) exist and are bounded by known constants [30]–[32].

Assumption 2. The time delay is bounded such that $0\leq\tau\,(t)\leq\varphi_1$ where $\varphi_1\in\mathbb{R}^+$ is a known constant and the rate of change of the delay is bounded such that $\|\dot{\tau}\,(t)\|<1$. Additionally, let $\ddot{\tau}\,(t)$ be bounded such that $\|\ddot{\tau}\,(t)\|\leq\varphi_2$ where $\varphi_2\in\mathbb{R}^+$ is a known constant.

The implications of Assumptions 1 and 2 are that the disturbance and delays are sufficiently smooth and that the delay is sufficiently slow. The development of an input-delayed controller for arbitrarily fast time-varying delays remains an open problem.

III. Control Objective

The objective is to design a continuous controller that will ensure the generalized state $q\,(t)$ of the input-delayed system in (1) tracks a desired trajectory despite uncertainties and additive bounded disturbances in the dynamic model. To quantify the control objective, a tracking error denoted by $e\,(q,t)\in\mathbb{R}^n$, is defined as

$$e\triangleq q_d-q \quad (2)$$

where $q_d\,(t)\in\mathbb{R}^n$ denotes the desired trajectory and is designed such that $q_d\,(t)\,,\dot{q}_d\,(t)\,,\ddot{q}_d\,(t)\in\mathcal{L}_\infty$. To facilitate the subsequent analysis, a measurable filtered tracking error, denoted by $r\,(e,e_z,t)\in\mathbb{R}^n$, is defined as

$$r\triangleq\dot{e}+\alpha e-Be_z \quad (3)$$

where $\alpha\in\mathbb{R}^+$ is a known gain constant, and $B\in\mathbb{R}^{n\times n}$ is a known symmetric, positive definite constant gain matrix that satisfies the following inequality

$$\|B\|_\infty\leq b \quad (4)$$

where $b\in\mathbb{R}^+$ is a known constant. In (3), $e_z\,(t)\in\mathbb{R}^n$ is an auxiliary signal containing the time-delays in the system, defined as

$$e_z\triangleq\int_{t-\tau(t)}^t u\,(\theta)\,\mathrm{d}\theta. \quad (5)$$

The error between B and $M^{-1}\,(q)$ is denoted by $\eta\,(q)\in\mathbb{R}^{n\times n}$ and is defined as

$$\eta\triangleq B-M^{-1} \quad (6)$$

and satisfies the following inequality

$$\|\eta\|_\infty\leq\bar{\eta} \quad (7)$$

where $\bar{\eta}\in\mathbb{R}^+$ is a known constant.

IV. Control Development

The open-loop error system can be obtained by multiplying the time derivative of (3) by $M\,(q)$ and utilizing the expressions in (1), (2), (5) and (6) to yield

$$\begin{aligned}M\dot{r}\;=\;&M\ddot{q}_d+V_m\dot{q}+G+F+d-M\eta\,(u-u_\tau+u_\tau\dot{\tau})\\&-u-u_\tau\dot{\tau}+\alpha M\dot{e}.\end{aligned} \quad (8)$$

Based on the error system formulation in (3) and (5), the open-loop error system in (8) contains a delay-free control input. From (8) and the subsequent stability analysis, the control input is designed as

$$u=k_br \quad (9)$$

where $k_b\in\mathbb{R}^+$ is a known constant control gain. To facilitate the subsequent stability analysis, an auxiliary signal, $N_d\,(q_d,\dot{q}_d,\ddot{q}_d)\in\mathbb{R}^n$, is defined as

$$N_d\triangleq M_d\ddot{q}_d+V_{md}\dot{q}_d+G_d+F_d$$

where M_d, V_{md}, G_d, F_d denote $M(q_d) \in \mathbb{R}^{n\times n}$, $V_m(q_d, \dot{q}_d) \in \mathbb{R}^{n\times n}$, $G(q_d) \in \mathbb{R}^n$, $F(\dot{q}_d) \in \mathbb{R}^n$, respectively.

The closed-loop error system is obtained by adding and subtracting $N_d(q_d, \dot{q}_d, \ddot{q}_d, t)$ and $e(t)$ to (8) and utilizing (3) and (9) to yield

$$\begin{aligned} M\dot{r} &= -\frac{1}{2}\dot{M}r + \chi + S - k_b r - k_b M\eta\left[r - r_\tau + r_\tau\dot{\tau}\right] \\ &\quad -k_b r_\tau \dot{\tau} - e \end{aligned} \tag{10}$$

where the auxiliary terms $\chi(e_1, r, e_z)$, $S(q_d, \dot{q}_d, \ddot{q}_d, t) \in \mathbb{R}^n$ are defined as

$$\begin{aligned} \chi &\triangleq \frac{1}{2}\dot{M}r + M\ddot{q}_d + V_m\dot{q} + G + F - N_d + \alpha M r \\ &\quad -\alpha^2 M e + \alpha M B e_z + e, \end{aligned} \tag{11}$$

$$S \triangleq N_d + d. \tag{12}$$

Using Assumption 1, the following inequality can be developed based on the expression in (12)

$$\|S\| \leq \bar{s} \tag{13}$$

where $\bar{s} \in \mathbb{R}^+$ is a known constant. The structure of (10) is motivated by the desire to segregate terms that can be upper bounded by state-dependent terms and terms that can be upper bounded by constants. Using the Mean Value Theorem, Property 1, and (4), the expression in (11) can be upper bounded as

$$\|\chi\| \leq \rho(\|z\|)\|z\| \tag{14}$$

where $\rho(\|z\|)$ is a positive globally invertible nondecreasing function and $z(e, r, e_z) \in \mathbb{R}^{3n}$ is defined as

$$z \triangleq \begin{bmatrix} e^T & r^T & e_z^T \end{bmatrix}^T. \tag{15}$$

To facilitate the subsequent stability analysis, let $y(e, r, P, Q) \in \mathbb{R}^{2n+2}$ be defined as

$$y \triangleq \begin{bmatrix} e^T & r^T & \sqrt{P} & \sqrt{Q} \end{bmatrix}^T \tag{16}$$

where $P(u, t, \tau), Q(r, t, \tau, \dot{\tau}) \in \mathbb{R}$ denote LK functionals defined as

$$P \triangleq \omega \int_{t-\tau(t)}^{t} \left(\int_{s}^{t} \|u(\theta)\|^2 \mathrm{d}\theta \right) \mathrm{d}s \tag{17}$$

$$Q \triangleq \frac{k_b(2\bar{m}\bar{\eta} + 1)}{2(1-\dot{\tau})} \int_{t-\tau(t)}^{t} \|r(\theta)\|^2 \mathrm{d}\theta \tag{18}$$

and $\omega \in \mathbb{R}^+$ is a known constant. Additionally, let k_b, introduced in (9), be defined as

$$k_b \triangleq k_1 + k_2 + k_3 \tag{19}$$

where $k_1, k_2, k_3 \in \mathbb{R}^+$.

Based on the result of the subsequent stability analysis, the control gains $\alpha, \gamma, k_1, k_2, k_3$ are selected according to the following sufficient conditions

$$\alpha > \frac{b^2\psi^2}{4}, \quad k_b > \sup_{\tau,\dot{\tau}} \left(\frac{\varphi_2(2\bar{m}\bar{\eta}+1)}{2(1-\dot{\tau})^2(\psi^2\omega(1-\dot{\tau}) + \tau)} \right),$$

$$k_3 > \sup_{\tau,\dot{\tau}} \left(k_b^2\omega\tau + \frac{2k_b\bar{m}\bar{\eta}}{1-\dot{\tau}} \right) \tag{20}$$

where $\psi \in \mathbb{R}^+$ is a known constant. Let the auxiliary constant $\beta \in \mathbb{R}^+$ be defined as

$$\begin{aligned} \beta = \inf_{\tau,\dot{\tau}} \Bigg[&\alpha - \frac{b^2\psi^2}{4},\ k_3 - k_b^2\omega\tau - \frac{k_b(2\bar{m}\bar{\eta}(2\dot{\tau}-3) + \dot{\tau} - 2)}{2(\dot{\tau}-1)}, \\ &\frac{1}{\tau}\left(\omega(1-\dot{\tau}) - \frac{\varphi_2(2\bar{m}\bar{\eta}+1)}{2k_b(1-\dot{\tau})^2} - \frac{2\tau}{\psi^2} \right) \Bigg]^T. \end{aligned} \tag{21}$$

If the sufficient conditions in (20) are satisfied and $\tau(t), \dot{\tau}(t)$ are sufficiently small, then $\beta > 0$.

V. Stability Analysis

Theorem 1. *Given the dynamics in (1), the controller in (9) ensures semi-global uniformly ultimately bounded tracking in the sense that*

$$\|e(t)\| \leq \epsilon_0 exp(-\epsilon_1 t) + \epsilon_2 \tag{22}$$

where $\epsilon_0, \epsilon_1, \epsilon_2 \in \mathbb{R}^+$ denote constants, provided the sufficient conditions in (20) are satisfied, and $\tau(t)$, $\dot{\tau}(t)$ are sufficiently small (see (20) and Assumption 2).

Proof: Let $V_L(y, t) : \mathcal{D} \times [0, \infty) \to \mathbb{R}$ be a continuously differentiable, positive-definite functional on a domain $\mathcal{D} \subseteq \mathbb{R}^{2n+2}$, defined as

$$V_L \triangleq \frac{1}{2}e^T e + \frac{1}{2}r^T M r + P + Q \tag{23}$$

which can be bounded as

$$\phi_1 \|y\|^2 \leq V_L \leq \phi_2 \|y\|^2$$

where the constants $\phi_1, \phi_2 \in \mathbb{R}$ are defined as

$$\phi_1 \triangleq \frac{1}{2}min\left[\underline{m}, 1\right], \quad \phi_2 \triangleq max\left[\frac{1}{2}\bar{m}, 1\right]. \tag{24}$$

After utilizing (3) and (10), applying the Leibniz Rule to determine the time derivative of (17) and (18), and by canceling similar terms, the time derivative of (23) can be expressed as

$$\begin{aligned} \dot{V}_L &= -\alpha e^T e - k_b r^T r \\ &\quad + B e^T e_z - k_b M\eta r^T r + k_b M\eta(1-\dot{\tau}) r^T r_\tau \\ &\quad - k_b\dot{\tau} r^T r_\tau + r^T\chi + r^T S \\ &\quad + \omega\tau\|u\|^2 - \omega(1-\dot{\tau})\int_{t-\tau(t)}^{t} \|u(\theta)\|^2 d\theta \\ &\quad + \frac{k_b(2\bar{m}\bar{\eta}+1)}{2(1-\dot{\tau})}\|r\|^2 - \frac{k_b(2\bar{m}\bar{\eta}+1)}{2}\|r_\tau\|^2 \\ &\quad + \frac{k_b(2\bar{m}\bar{\eta}+1)\ddot{\tau}}{2(1-\dot{\tau})^2}\int_{t-\tau(t)}^{t} \|r(\theta)\|^2 \mathrm{d}\theta. \end{aligned} \tag{25}$$

By utilizing Assumption 2, (4), (7), (9), (13), and (14), (25) can be expanded, regrouped and upper bounded as

$$\begin{aligned}\dot{V}_L \quad \leq \quad & -\alpha\left\|e\right\|^2 - k_b\left\|r\right\|^2 + b\left\|e\right\|\left\|e_z\right\| + k_b\bar{m}\bar{\eta}\left\|r\right\|^2 \\ & + k_b\left(2\bar{m}\bar{\eta}+1\right)\left\|r\right\|\left\|r_\tau\right\| \\ & + \rho\left(\left\|z\right\|\right)\left\|z\right\|\left\|r\right\| + \left\|r\right\|\bar{s} \\ & + k_b^2\omega\tau\left\|r\right\|^2 - \omega\left(1-\dot{\tau}\right)\int_{t-\tau(t)}^{t}\left\|u\left(\theta\right)\right\|^2 d\theta \\ & + \frac{k_b\left(2\bar{m}\bar{\eta}+1\right)}{2\left(1-\dot{\tau}\right)}\left\|r\right\|^2 - \frac{k_b\left(2\bar{m}\bar{\eta}+1\right)}{2}\left\|r_\tau\right\|^2 \\ & + \frac{\varphi_2\left(2\bar{m}\bar{\eta}+1\right)}{2k_b\left(1-\dot{\tau}\right)^2}\int_{t-\tau(t)}^{t}\left\|u\left(\theta\right)\right\|^2 \mathrm{d}\theta. \qquad (26)\end{aligned}$$

Young's Inequality can be used to upper bound select terms in (26) as

$$\begin{aligned} b\left\|e\right\|\left\|e_z\right\| \quad & \leq \quad \frac{b^2\psi^2}{4}\left\|e\right\|^2 + \frac{1}{\psi^2}\left\|e_z\right\|^2, \\ \left\|r\right\|\left\|r_\tau\right\| \quad & \leq \quad \frac{1}{2}\left\|r\right\|^2 + \frac{1}{2}\left\|r_\tau\right\|^2 \qquad (27)\end{aligned}$$

where ψ was introduced in (20). Using (27), (26) can be upper bounded as

$$\begin{aligned}\dot{V}_L \leq & -\alpha\left\|e\right\|^2 - k_b\left\|r\right\|^2 + \frac{b^2\psi^2}{4}\left\|e\right\|^2 \\ & + \frac{\tau}{\psi^2}\int_{t-\tau(t)}^{t}\left\|u\left(\theta\right)\right\|^2 \mathrm{d}\theta + k_b\bar{m}\bar{\eta}\left\|r\right\|^2 \qquad (28) \\ & + \frac{k_b\left(2\bar{m}\bar{\eta}+1\right)}{2}\left\|r\right\|^2 + \frac{k_b\left(2\bar{m}\bar{\eta}+1\right)}{2\left(1-\dot{\tau}\right)}\left\|r\right\|^2 \\ & + \rho\left(\left\|z\right\|\right)\left\|z\right\|\left\|r\right\| + \left\|r\right\|\bar{s} + k_b^2\omega\tau\left\|r\right\|^2 \\ & - \left(\omega\left(1-\dot{\tau}\right) - \frac{\varphi_2\left(2\bar{m}\bar{\eta}+1\right)}{2k_b\left(1-\dot{\tau}\right)^2}\right)\int_{t-\tau(t)}^{t}\left\|u\left(\theta\right)\right\|^2 d\theta.\end{aligned}$$

Utilizing the Cauchy-Schwarz inequality and (5) yields

$$\left\|e_z\right\|^2 \leq \tau\int_{t-\tau(t)}^{t}\left\|u\left(\theta\right)\right\|^2 \mathrm{d}\theta. \qquad (29)$$

After adding and subtracting $\frac{\tau}{\psi^2}\int_{t-\tau}^{t}\left\|u\left(\theta\right)\right\|^2 \mathrm{d}\theta$, utilizing (20) and (29), and canceling terms, (28) can be expressed as

$$\begin{aligned}\dot{V}_L \leq & -\alpha\left\|e\right\|^2 - k_b\left\|r\right\|^2 \\ & + \frac{b^2\psi^2}{4}\left\|e\right\|^2 + k_b\bar{m}\bar{\eta}\left\|r\right\|^2 + \frac{k_b\left(2\bar{m}\bar{\eta}+1\right)}{2}\left\|r\right\|^2 \\ & + \rho\left(\left\|z\right\|\right)\left\|z\right\|\left\|r\right\| + \left\|r\right\|\bar{s} + k_b^2\omega\tau\left\|r\right\|^2 \\ & - \frac{1}{\tau}\left(\omega\left(1-\dot{\tau}\right) - \frac{\varphi_2\left(2\bar{m}\bar{\eta}+1\right)}{2k_b\left(1-\dot{\tau}\right)^2} - \frac{2\tau}{\psi^2}\right)\left\|e_z\right\|^2 \\ & + \frac{k_b\left(2\bar{m}\bar{\eta}+1\right)}{2\left(1-\dot{\tau}\right)}\left\|r\right\|^2 - \frac{\tau}{\psi^2}\int_{t-\tau(t)}^{t}\left\|u\left(\theta\right)\right\|^2 \mathrm{d}\theta. \ (30)\end{aligned}$$

Inserting k_1, k_2, k_3 from (19) and regrouping terms, (30) can be upper bounded as

$$\begin{aligned}\dot{V}_L \ \leq \ & -\left(\alpha - \frac{b^2\psi^2}{4}\right)\left\|e\right\|^2 \\ & - k_1\left\|r\right\|^2 + \rho\left(\left\|z\right\|\right)\left\|z\right\|\left\|r\right\| - k_2\left\|r\right\|^2 + \left\|r\right\|\bar{s} \\ & - \left(k_3 - k_b\bar{m}\bar{\eta} - k_b^2\omega\tau\right)\left\|r\right\|^2 \\ & + \frac{k_b\left(2\bar{m}\bar{\eta}+1\right)}{2}\left(1 + \frac{1}{\left(1-\dot{\tau}\right)}\right)\left\|r\right\|^2 \\ & - \frac{1}{\tau}\left(\omega\left(1-\dot{\tau}\right) - \frac{\varphi_2\left(2\bar{m}\bar{\eta}+1\right)}{2k_b\left(1-\dot{\tau}\right)^2} - \frac{2\tau}{\psi^2}\right)\left\|e_z\right\|^2 \\ & - \frac{\tau}{\psi^2}\int_{t-\tau(t)}^{t}\left\|u\left(\theta\right)\right\|^2 \mathrm{d}\theta. \qquad (31)\end{aligned}$$

After completing the squares, the expression in (31) can be upper bounded as

$$\begin{aligned}\dot{V}_L \ \leq \ & -\left(\beta - \frac{\rho^2\left(\left\|z\right\|\right)}{4k_1}\right)\left\|z\right\|^2 \\ & - \frac{\tau}{\psi^2}\int_{t-\tau(t)}^{t}\left\|u\left(\theta\right)\right\|^2 \mathrm{d}\theta + \frac{\bar{s}^2}{4k_2} \qquad (32)\end{aligned}$$

where β was defined in (21). The inequality [19]

$$\begin{aligned}& \int_{t-\tau(t)}^{t}\left(\int_{s}^{t}\left\|u\left(\theta\right)\right\|^2 \mathrm{d}\theta\right)\mathrm{d}s \leq \\ & \tau\sup_{s\in[t,\,t-\tau]}\left[\int_{s}^{t}\left\|u\left(\theta\right)\right\|^2 \mathrm{d}\theta\right] = \tau\int_{t-\tau(t)}^{t}\left\|u\left(\theta\right)\right\|^2 \mathrm{d}\theta\end{aligned}$$

can be used to upper bound (32) as

$$\begin{aligned}\dot{V}_L \leq & -\left(\beta - \frac{\rho^2\left(\left\|z\right\|\right)}{4k_1}\right)\left\|z\right\|^2 + \frac{\bar{s}^2}{4k_2} \qquad (33) \\ & - \frac{1}{2\psi^2}\int_{t-\tau(t)}^{t}\left(\int_{s}^{t}\left\|u\left(\theta\right)\right\|^2 \mathrm{d}\theta\right) - \frac{\tau}{2\psi^2}\int_{t-\tau(t)}^{t}\left\|u\left(\theta\right)\right\|^2 \mathrm{d}\theta.\end{aligned}$$

Using the definitions of $u\left(r,t\right)$ in (9), $z\left(t\right)$ in (15), and $y\left(t\right)$ in (16), the expression in (33) can be upper bounded as

$$\dot{V}_L \leq -\beta_2\left\|y\right\|^2 - \left(\beta - \frac{\rho^2\left(\left\|z\right\|\right)}{4k_1}\right)\left\|e_z\right\|^2 + \frac{\bar{s}^2}{4k_2} \qquad (34)$$

where $\beta_2\left(\left\|z\right\|\right) \in \mathbb{R}^+$ is defined as

$$\beta_2 = \underset{\tau,\dot{\tau}}{inf}\left[\beta - \frac{\rho^2\left(\left\|z\right\|\right)}{4k_1},\ \frac{1}{2\omega\psi^2},\ \frac{k_b\tau\left(1-\dot{\tau}\right)}{\psi^2\left(2\bar{m}\bar{\eta}+1\right)}\right]^T.$$

By further utilizing (24), the inequality in (34) can be written as

$$\dot{V}_L \leq -\frac{\beta_2}{\phi_2}V_L + \frac{\bar{s}^2}{4k_2}. \qquad (35)$$

Consider a set S defined as

$$S \triangleq \left\{z\left(t\right) \in \mathbb{R}^{3n} \mid \left\|z\right\| < \rho^{-1}\left(2\sqrt{\beta k_1}\right)\right\}.$$

In S, $\beta_2\left(\left\|z\right\|\right)$ can be lower bounded by a constant $\delta \in \mathbb{R}^+$ as $\delta \leq \beta_2\left(\left\|z\right\|\right)$. Thus, the linear differential equation in (35) can be solved as

$$V_L \leq V_L\left(0\right)e^{-\frac{\delta}{\phi_2}t} + \frac{\bar{s}^2\phi_2}{4k_2\delta}\left[1 - e^{-\frac{\delta}{\phi_2}t}\right], \qquad (36)$$

Time-Delay	$\tau(t)$ (ms)	RMS Errors (deg)	
		Link 1	Link 2
Fast, Small	$2 \cdot sin\left(\frac{t}{2}\right)+3$	0.0524^o	0.0363^o
Fast, Large	$20 \cdot sin\left(\frac{t}{2}\right)+30$	0.4913^o	0.5687^o
Slow, Small	$2 \cdot sin\left(\frac{t}{10}\right)+3$	0.0521^o	0.0341^o
Slow, Large	$20 \cdot sin\left(\frac{t}{10}\right)+30$	0.5179^o	0.6970^o

Table I
RMS ERRORS FOR TIME-VARYING TIME-DELAY RATES AND MAGNITUDES.

provided $\|z\| < \rho^{-1}\left(2\sqrt{\beta k_1}\right)$. From (36), given $z(0)$, k_1 can be selected such that $z(0) \in S$ (i.e. a semi-global result) to yield the result in (22). ■

VI. Simulation Results

The controller developed in (9) was simulated for a two-link planar manipulator. The dynamics of the manipulator are given as

$$\begin{aligned}\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} &= \begin{bmatrix} p_1+2p_3c_2 & p_2+p_3c_2\\ p_2+p_3c_2 & p_2\end{bmatrix}\begin{bmatrix}\ddot{q}_1\\ \ddot{q}_2\end{bmatrix}\\ &+\begin{bmatrix} -p_3s_2\dot{q}_2 & -p_3s_2\left(\dot{q}_1+\dot{q}_2\right)\\ p_3s_2\dot{q}_1 & 0\end{bmatrix}\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\end{bmatrix}\\ &+\begin{bmatrix} f_{d1} & 0\\ 0 & f_{d2}\end{bmatrix}\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\end{bmatrix}+\begin{bmatrix}\tau_{d1}\\ \tau_{d2}\end{bmatrix}\end{aligned}$$

where $p_1 = 3.473\ kg\cdot m^2$, $p_2 = 0.196\ kg\cdot m^2$, $p_3 = 0.242\ kg\cdot m^2$, $f_{d1} = 5.3\ Nm\ sec$, $f_{d2} = 1{,}1\ Nm\ sec$, c_2 denotes $cos\left(q_2\right)$, and s_2 denotes $sin\left(q_2\right)$. An additive exogenous disturbance was applied as $\tau_{d1} = 0.2sin\left(\frac{t}{2}\right)$, and $\tau_{d2} = 0.1sin\left(\frac{t}{4}\right)$. The desired trajectories for links 1 and 2 for all simulations were selected as

$$\begin{aligned} q_{d1}(t) &= 20sin\left(1.5t\right)\left(1-e^{-0.01t^3}\right)\ deg,\\ q_{d2}(t) &= 10sin\left(1.5t\right)\left(1-e^{-0.01t^3}\right)\ deg.\end{aligned}$$

The initial conditions for the manipulator were selected as stationary at $q_1, q_2 = 0\ deg$. Because the controller in (9) assumes that the inertia matrix is unknown, a best guess estimate of the constant matrix B is selected as

$$B = \begin{bmatrix} 4.0 & 0.4\\ 0.4 & 0.2\end{bmatrix}.$$

To illustrate robustness to the input delay, simulations were completed using various time-varying delays. For each case, the root mean square (RMS) errors are shown in Table I. The results show that the performance of the system appears to be independent of delay frequency but is affected when the delay magnitude increases. This outcome agrees with previous input delay results which showed that tracking performance is reduced as larger constant time-delays are applied to the system [19].

Analysis was also conducted to examine the robustness of the controller with respect to unknown variances in the the time-delay. In each case, the input delay entering the plant was varied from the delay used in the controller feedback.

Time-Delay Variance in Plant	RMS Errors (deg)	
	Link 1	Link 2
-30% magnitude	0.0633^o	0.0766^o
-10% magnitude	0.0497^o	0.0662^o
0% magnitude	0.0394^o	0.0605^o
+10% magnitude	0.0495^o	0.0764^o
+30% magnitude	0.0628^o	0.1069^o
+10% phase	0.0393^o	0.0605^o
+30% phase	0.0394^o	0.0604^o
+50% phase	0.0405^o	0.0619^o

Table II
RMS ERRORS FOR CASES OF UNCERTAINTY IN TIME-VARYING TIME-DELAY SEEN BY THE PLANT AS COMPARED TO THE DELAY ASSUMED BY THE CONTROLLER.

The controller assumes a sinusoidal time-delay with a peak magnitude of $10\ ms$. The results are presented in Table II. The results suggest that the controller is robust to variances in delay magnitude and phase shift. Specifically, the uncertainty in phase of the delay exhibits negligible degradation on the tracking performance. Figure 1 illustrates the time-delay and the tracking errors associated with the +50% phase variance case.

Additional results show that the performance/robustness of the developed controller with respect to the mismatch between B and $M^{-1}(q)$ indicates an insignificant amount of variation in the performance even when each element of $M^{-1}(q)$ is overestimated by as much as 100%.

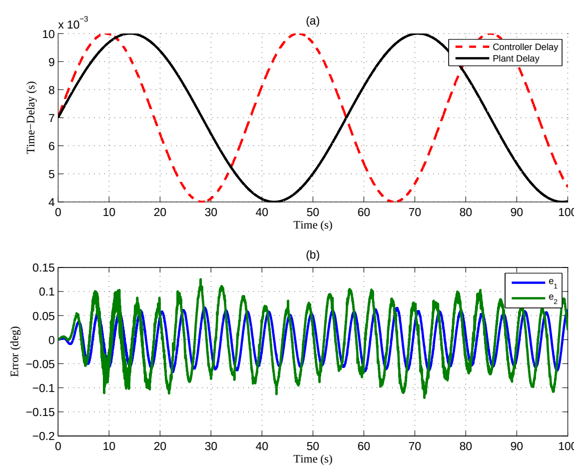

Figure 1. +50% phase variance in input-delay with peak magnitude of 10 ms. (a) Time-delay in seconds vs time. (b) Tracking error in degrees vs time.

VII. Conclusion

A continuous controller is developed for uncertain nonlinear Euler-Lagrange systems which include time-varying input delays and additive bounded disturbances. The controller assumes that the time-delay is bounded and slowly varying and is shown to guarantee uniformly ultimately bounded tracking in the presence of model uncertainty and/or unmodeled effects. Numerical simulations demonstrate the robustness of the control design with respect to delay variation and uncertainty. Extending the result to include uncertain time-delays will enhance the applicability of the controller, and is the focus of on-going efforts.

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