RISE-Based Control of an Uncertain Nonlinear System With Time-Varying State Delays

N. Fischer, R. Kamalapurkar, N. Sharma, W. E. Dixon

***Abstract*—This paper considers a continuous control design for second-order control affine nonlinear systems with time-varying state delays. A neural network is augmented with a robust integral of the sign of the error (RISE) control structure to achieve semi-global asymptotic tracking in the presence of unknown, arbitrarily large, time-varying delays, not linear-in-the-parameters uncertainty and additive bounded disturbances. By expressing unknown functions in terms of the desired trajectories and through strategic grouping of delay-free and delay-dependent terms, Lyapunov-Krasovskii functionals are utilized to cancel the delayed terms in the analysis and obtain delay-free neural network update laws.**

I. Introduction

Stability and control of dynamical systems with time-delays in the state and/or control has received considerable attention for more than four decades [1]–[4]. Motivated by performance and stability problems with time-delayed systems, solutions typically use appropriate Lyapunov-Razumikhin or Lyapunov-Krasovskii functionals to derive bounds on the delay such that the closed loop system is stable. Numerous methods have been developed throughout literature for time-delayed linear systems and nonlinear systems with known dynamics [1], [4]–[7]. For uncertain nonlinear systems, techniques have also been developed to compensate for both known and unknown constant state-delays [8]–[15]. Extensions of these designs to systems with nonlinear, bounded disturbances also exist [13], [15], [16].

For some applications, it is often more practical to consider time-varying or state-dependent time-delays. Control methods for uncertain nonlinear systems with time-varying state delays have been studied in results such as [12], [17]–[20]. However, compensation of time-varying state-delays in systems with both uncertain dynamics and added exogenous disturbances is explored in only a few results. A robust integral sliding mode technique for stochastic systems with time-varying delays and linearly state-bounded nonlinear uncertainties is developed in [21] but depends on convex optimization routines and an LMI feasibility condition. In [22], an adaptive fuzzy logic control method yielding a semi-global uniformly ultimately bounded tracking result is illustrated for a SISO system in Brunovsky form. The authors of [23] utilize the circle criterion and an LMI feasibility condition to design a nonlinear observer for neural-network-based control of a class of uncertain stochastic nonlinear strict-feedback systems. The design proposes a neural network weight update law that directly cancels the bound on the reconstruction error to yield a globally stable result. Discontinuous model reference adaptive controllers have been designed in [24] and [25] for uncertain nonlinear plants with time-varying delays to achieve asymptotic stability results; however, the discontinuous nature of these results motivates the design of continuous control techniques.

In this paper, a continuous controller for uncertain nonlinear systems with an unknown, arbitrarily large, time-varying state delay is developed. Motivated by our previous work in [26], a continuous robust integral of the sign of the error (RISE) control structure is augmented with a three-layer neural network (NN) to compensate for time-varying state delays which are arguments of uncertain nonautonomous functions that contain not linear-in-the-parameters (non-LP) uncertainty. Under the assumption that the time-delay can be arbitrarily large, bounded and slowly varying, Lyapunov-Krasovskii (LK) functionals are utilized to prove semi-global asymptotic tracking. In comparison to our previous work for constant state delays in [27], new efforts in this paper required to compensate for time-varying state delays include: strategic grouping of delay-dependent and delay-free terms and a redesigned LK functional. In comparison to [27], neural networks are used in the current work to compensate for the non-LP disturbances, and new efforts are required to design the online NN update laws in the presence of the unknown time-varying delay.

II. Dynamic Model and Properties

Consider a class of uncertain second-order control affine nonlinear systems with an unknown time-varying state delay described by

$$\ddot{x} = f\left(x,\dot{x},t\right)+g\left(x\left(t-\tau\right),\dot{x}\left(t-\tau\right),t\right)+d\left(t\right)+u\left(t\right). \quad (1)$$

In (1), $f\left(x,\dot{x},t\right):\mathbb{R}^{2n}\times[0,\infty)\rightarrow\mathbb{R}^{n}$ is an unknown function, $g\left(x\left(t-\tau\right),\dot{x}\left(t-\tau\right),t\right):\mathbb{R}^{2n}\times[0,\infty)\rightarrow\mathbb{R}^{n}$ is an unknown time-delayed function, $\tau\left(t\right)\in\mathbb{R}$ is an unknown, time-varying, arbitrarily large time-delay, $d\left(t\right):[0,\infty)\rightarrow\mathbb{R}^{n}$ is a sufficiently smooth bounded disturbance (e.g., unmodeled effects), $u\left(t\right)\in\mathbb{R}^{n}$ is the control input, and $x\left(t\right),\dot{x}\left(t\right)\in\mathbb{R}^{n}$ are measurable system states. Throughout

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the paper, a time-dependent delayed function is denoted as $\zeta(t-\tau)$ or ζ_τ, and $\|\cdot\|$ denotes the Euclidean norm of a vector. Additionally, the following assumptions are used.

Assumption 1. The unknown time delay is bounded such that $0 \le \tau(t) \le \varphi_1$ and the rate of change of the delay is bounded such that $|\dot{\tau}(t)| \le \varphi_2 < 1$ where $\varphi_1, \varphi_2 \in \mathbb{R}^+$ are known constants.

Assumption 2. The functions $f(\cdot), g(\cdot)$ and their first and second derivatives with respect to their arguments are Lipschitz continuous.

Assumption 3. The nonlinear disturbance term and its first two time derivatives (i.e., $d(t), \dot{d}(t), \ddot{d}(t)$) exist and are bounded by known constants [26]–[28].

Assumption 4. The desired trajectory is designed such that $x_d^{(i)}(t) \in \mathbb{R}^n$, $\forall i = 0, 1, ..., 4$ exist and are bounded.[1]

III. Control Development

The control objective is to design a continuous controller that will ensure $x(t)$ tracks a desired trajectory. To quantify the control objective, a tracking error denoted $e_1(x,t) \in \mathbb{R}^n$ is defined as

$$e_1 \triangleq x_d - x. \tag{2}$$

To facilitate the subsequent analysis, two filtered tracking errors, denoted by $e_2(e_1, \dot{e}_1, t), r(e_2, \dot{e}_2, t) \in \mathbb{R}^n$, are defined as

$$e_2 \triangleq \dot{e}_1 + \alpha_1 e_1 \tag{3}$$
$$r \triangleq \dot{e}_2 + \alpha_2 e_2 \tag{4}$$

where $\alpha_1, \alpha_2 \in \mathbb{R}^+$ are known gain constants. The auxiliary signal $r(e_2, \dot{e}_2, t)$ is introduced to facilitate the stability analysis and is not used in the control design since the expression in (4) depends on the unmeasurable state $\ddot{x}(t)$.

An open-loop tracking error can be obtained by substituting for (1)-(4) to yield

$$\begin{aligned} r = {} & \alpha_1 e_1 + \alpha_2 e_2 + \ddot{x}_d - d \\ & - f(x, \dot{x}, t) - g(x_\tau, \dot{x}_\tau, t) - u. \end{aligned} \tag{5}$$

Using a desired compensation adaptation law (DCAL)-based design approach [29], (5) can be written as

$$\begin{aligned} r = {} & \alpha_1 e_1 + \alpha_2 e_2 + S_1 + S_d + \ddot{x}_d - d \\ & + g(x_d, \dot{x}_d) - g(x_{d\tau}, \dot{x}_{d\tau}) - u \end{aligned} \tag{6}$$

where the auxiliary functions $S_1(x, x_d, \dot{x}, \dot{x}_d, x_\tau, \dot{x}_\tau, x_{d\tau}, \dot{x}_{d\tau}, t)$, $S_d(x_d, \dot{x}_d) \in \mathbb{R}^n$ are defined as

$$S_1 \triangleq -f(x, \dot{x}, t) + f(x_d, \dot{x}_d) - g(x_\tau, \dot{x}_\tau, t) + g(x_{d\tau}, \dot{x}_{d\tau}),$$
$$S_d \triangleq -f(x_d, \dot{x}_d) - g(x_d, \dot{x}_d).$$

The grouping of terms in (5) is motivated by the desire to segregate terms that can be upper bounded by state-dependent terms (whether delayed or delay-free) from the terms that can be upper bounded by constants.

The Universal Approximation Theorem can be used to represent the auxiliary function $S_d(\cdot)$ by a three-layer NN as

$$S_d \triangleq W^T \sigma\left(V^T x_{nn}\right) + \varepsilon \tag{7}$$

where $V(t) \in \mathbb{R}^{(N_1+1)\times N_2}$ and $W(t) \in \mathbb{R}^{(N_2+1)\times n}$ are bounded constant ideal weights for the first-to-second and second-to-third layers, respectively, N_1 is the number of neurons in the input layer, N_2 is the number of neurons in the hidden layer, n is the number of neurons in the output layer, $\sigma(\cdot) \in \mathbb{R}^{N_2+1}$ is an activation function, $x_{nn}(t) \in \mathbb{R}^{N_1+1}$ denotes the input to the NN defined on a compact set containing the known bounded desired trajectories as $x_{nn} = \left[1, x_d^T, \dot{x}_d^T\right]^T$, and $\varepsilon(x_{nn}) \in \mathbb{R}^n$ denotes the functional reconstruction errors.

Assumption 5. The ideal NN weights are assumed to exist and be bounded by known positive constants, i.e. $\|V\|_F^2 \le V_B$, $\|W\|_F^2 \le W_B$ where $\|\cdot\|_F$ is the Frobenius norm for a matrix.

Assumption 6. The functional reconstruction errors $\varepsilon(\cdot)$ and their first derivative with respect to their arguments are bounded such that $\|\varepsilon(x_{nn})\| \le \varepsilon_{b1}$, $\|\dot{\varepsilon}(x_{nn}, \dot{x}_{nn})\| \le \varepsilon_{b2}$, where $\varepsilon_{b_1}, \varepsilon_{b_2} \in \mathbb{R}$ are known positive constants.

Assumption 7. The activation function $\sigma(\cdot)$ and its derivative, $\sigma'(\cdot)$ are bounded.

Remark 1. Assumptions 5-6 are standard assumptions in NN control literature (cf. [30]). The ideal weight upper bounds are assumed to be known to facilitate the use of the projection algorithm to ensure the weight estimates are always bounded. There are numerous activations functions which satisfy Assumption 7, e.g., sigmoidal or hyperbolic tangent functions.

The controller is designed using a three-layer NN feedforward term augmented by a RISE feedback term as

$$u \triangleq \hat{S}_d + \mu. \tag{8}$$

The RISE feedback term $\mu(e_2, \upsilon) \in \mathbb{R}^n$ is defined as [31], [32]

$$\mu \triangleq (k_s + 1) e_2 - (k_s + 1) e_2(0) + \upsilon \tag{9}$$

where $\upsilon(e_2) \in \mathbb{R}^n$ is the generalized Filippov solution to the following differential equation

$$\dot{\upsilon} \triangleq (k_s + 1) \alpha_2 e_2 + \beta sgn(e_2), \tag{10}$$

$\beta, k_s \in \mathbb{R}$ are positive, constant control gains, and $sgn(\cdot)$ is defined $\forall \xi \in \mathbb{R}^n = \begin{bmatrix} \xi_1 & \xi_2 & \ldots & \xi_n \end{bmatrix}^T$ as $sgn(\xi) \triangleq \begin{bmatrix} sgn(\xi_1) & sgn(\xi_2) & \ldots & sgn(\xi_n) \end{bmatrix}^T$.[2]

[1] Many guidance and navigation applications utilize smooth, high-order differentiable desired trajectories. Curve fitting methods can also be used to generate sufficiently smooth time-varying trajectories.

[2] The initial condition for $\upsilon(0)$ is selected such that $u(0) = 0$.

Using Filippov's theory of differential inclusions [33]–[36], the existence of solutions can be established for $\dot{\upsilon} \in K[h_1](e_2)$, where $h_1(e_2) \in \mathbb{R}^n$ is defined as the right-hand side of $\dot{\upsilon}$ in (10) and $K[h_1] \triangleq \bigcap_{\delta>0} \bigcap_{\mu S_m=0} \overline{co} h_1(B(e_2,\delta) - S_m)$, where $\bigcap_{\mu S_m=0}$ denotes the intersection over all sets S_m of Lebesgue measure zero, $\overline{co}$ denotes convex closure, and $B(e_2,\delta) = \{\varsigma \in \mathbb{R}^n | \, \|e_2 - \varsigma\| < \delta\}$ [37], [38]. The differential equation given in (10) is continuous except for the Lebesgue measure zero set of times $t \in [t_0, t_f]$ when $e_2(e_1, \dot{e}_1, t) = 0$. Hence, the set of time-instances for which $\dot{\upsilon}(e_2)$ is not defined is Lebesgue negligible. The absolutely continuous solution $\upsilon(e_2) = \upsilon(e_2(t_0)) + \int_{t_0}^{t} \dot{\upsilon} dt$ does not depend on the value of $\dot{\upsilon}$ on a Lebesgue negligible set of time-instances [39].

The NN feedforward term $\hat{S}_d(t) \in \mathbb{R}^n$ in (8) is designed as

$$\hat{S}_d \triangleq \hat{W}^T \sigma\left(\hat{V}^T x_{nn}\right) \tag{11}$$

where $\hat{V}(t) \in \mathbb{R}^{(N_1+1)\times N_2}$ and $\hat{W}(t) \in \mathbb{R}^{(N_2+1)\times n}$ are estimates of the ideal weights. Based on the subsequent stability analysis, the DCAL-based weight update laws for the NN in (11) are generated online as

$$\dot{\hat{W}} \triangleq proj\left(\Gamma_1 \hat{\sigma}' \hat{V}^T \dot{x}_{nn} e_2^T\right) \tag{12}$$

$$\dot{\hat{V}} \triangleq proj\left(\Gamma_2 \dot{x}_{nn}\left(\hat{\sigma}'^T \hat{W} e_2\right)^T\right), \tag{13}$$

where $\Gamma_1 \in \mathbb{R}^{(N_2+1)\times(N_2+1)}$ and $\Gamma_2 \in \mathbb{R}^{(N_1+1)\times(N_1+1)}$ are positive-definite, constant symmetric control gain matrices, and $\hat{\sigma}'(\cdot) \in \mathbb{R}^{N_2+1}$ denotes the partial derivative of $\hat{\sigma} \triangleq \sigma\left(\hat{V}^T x_{nn}\right)$.

The closed-loop dynamics are developed by substituting (8)-(11) into (6), taking the time derivative, and adding and subtracting $W^T \hat{\sigma}' \hat{V}^T \dot{x}_{nn} + \hat{W}^T \hat{\sigma}' \tilde{V}^T \dot{x}_{nn}$ to yield

$$\begin{aligned}\dot{r} = {} & \alpha_1 \dot{e}_1 + \alpha_2 \dot{e}_2 + \dot{S}_1 + \dddot{x}_d - \dot{d} \\ & -\dot{g}(x_{d\tau}, \dot{x}_{d\tau}, \ddot{x}_{d\tau}) + \dot{g}(x_d, \dot{x}_d, \ddot{x}_d) \\ & -(k_s+1) r - \beta sgn(e_2) + \hat{W}^T \hat{\sigma}' \tilde{V}^T \dot{x}_{nn} \\ & + \tilde{W}^T \hat{\sigma}' \hat{V}^T \dot{x}_{nn} + W^T \sigma' V^T \dot{x}_{nn} - W^T \hat{\sigma}' \hat{V}^T \dot{x}_{nn} \\ & - \hat{W}^T \hat{\sigma}' \tilde{V}^T \dot{x}_{nn} - \dot{\hat{W}}^T \hat{\sigma} - \hat{W}^T \hat{\sigma}' \dot{\hat{V}}^T x_{nn} + \dot{\varepsilon}\end{aligned} \tag{14}$$

where estimate mismatches for the ideal weights, denoted $\tilde{V}(t) \in \mathbb{R}^{(N_1+1)\times N_2}$ and $\tilde{W}(t) \in \mathbb{R}^{(N_2+1)\times n}$, are defined as $\tilde{V}(t) = V(t) - \hat{V}(t)$ and $\tilde{W}(t) = W(t) - \hat{W}(t)$. Using the NN weight update laws from (12) and (13), the expression in (14) can be rewritten as

$$\dot{r} = \tilde{N} + N + e_2 - (k_s+1) r - \beta sgn(e_2) \tag{15}$$

where $\tilde{N}\left(\hat{W}, \hat{V}, e_1, e_2, \dot{e}_1, \dot{e}_2, t\right) \in \mathbb{R}^n$ and $N\left(\hat{W}, \hat{V}, t\right) \in \mathbb{R}^n$ are defined as

$$\begin{aligned}\tilde{N} \triangleq {} & \alpha_1 \dot{e}_1 + \alpha_2 \dot{e}_2 + \dot{S}_1 - e_2 - proj\left(\Gamma_1 \hat{\sigma}' \hat{V}^T \dot{x}_{nn} e_2^T\right)^T \hat{\sigma} \\ & - \hat{W}^T \hat{\sigma}' proj\left(\Gamma_2 \dot{x}_{nn}\left(\hat{\sigma}'^T \hat{W} e_2\right)^T\right),\end{aligned} \tag{16}$$

$$N \triangleq N_D + N_B. \tag{17}$$

In (17), $N_D(x_d, \dot{x}_d, \ddot{x}_d, \dddot{x}_d, t) \in \mathbb{R}^n$ is defined as

$$\begin{aligned}N_D \triangleq {} & W^T \sigma' V^T \dot{x}_{nn} + \dot{\varepsilon} + \dddot{x}_d - \dot{d} \\ & -\dot{g}(x_{d\tau}, \dot{x}_{d\tau}, \ddot{x}_{d\tau}) + \dot{g}(x_d, \dot{x}_d, \ddot{x}_d)\end{aligned} \tag{18}$$

and $N_B\left(\hat{W}, \hat{V}, x_d, \dot{x}_d, \ddot{x}_d, t\right) \in \mathbb{R}^n$ is separated such that

$$N_B \triangleq N_{B_1} + N_{B_2} \tag{19}$$

where $N_{B_1}\left(\hat{W}, \hat{V}, x_d, \dot{x}_d, \ddot{x}_d, t\right)$, $N_{B_2}\left(\hat{W}, \hat{V}, x_d, \dot{x}_d, \ddot{x}_d, t\right) \in \mathbb{R}^n$ are defined as

$$N_{B_1} \triangleq -W^T \hat{\sigma}' \hat{V}^T \dot{x}_{nn} - \hat{W}^T \hat{\sigma}' \tilde{V}^T \dot{x}_{nn},$$

$$N_{B_2} \triangleq \hat{W}^T \hat{\sigma}' \tilde{V}^T \dot{x}_{nn} + \tilde{W}^T \hat{\sigma}' \hat{V}^T \dot{x}_{nn}.$$

Separating the terms in (17) is motivated by the fact that the different components have different bounds [26].

Using Assumptions 3-7, $N_D(\cdot)$ from (18) and $N_B(\cdot)$ from (19) and their time derivatives can be upper bounded as

$$\|N_D\| \leq \zeta_1, \|N_B\| \leq \zeta_2, \left\|\dot{N}_D\right\| \leq \zeta_3, \left\|\dot{N}_B\right\| \leq \zeta_4 + \zeta_5 \|e_2\|,$$

where $\zeta_i \in \mathbb{R}^+$, $\forall i = 1, ..., 5$ are known constants. Additionally, $\tilde{N}(\cdot)$ from (16) can be upper bounded as

$$\left\|\tilde{N}\right\| \leq \rho_1(\|z\|)\|z\| + \rho_2(\|z_\tau\|)\|z_\tau\| \tag{20}$$

where $z(e_1, e_2, r) \in \mathbb{R}^{3n}$ denotes the vector $z = \begin{bmatrix} e_1^T & e_2^T & r^T \end{bmatrix}^T$ and $\rho_1(\cdot), \rho_2(\cdot) : \mathbb{R} \to \mathbb{R}$ are positive, globally invertible functions. The upper bound for the auxiliary function $\tilde{N}(\cdot)$ is segregated into delay-free and delay-dependent bounding functions to eliminate the delayed terms with the use of an LK functional in the stability analysis. Specifically, let $R_{LK}(z,t) \in \mathbb{R}$ denote an LK functional defined as

$$R_{LK} \triangleq \frac{\gamma}{2k_s} \int_{t-\tau(t)}^{t} \rho_2^2(\|z(\sigma)\|)\|z(\sigma)\|^2 d\sigma \tag{21}$$

where $\gamma \in \mathbb{R}^+$ is an adjustable constant, and k_s and $\rho_2(\cdot)$ were introduced in (9) and (20), respectively.

IV. Stability Analysis

Theorem 1. *The controller proposed in (8) and the weight update laws designed in (12)-(13) ensure that the states and controller are bounded and the tracking errors are regulated in the sense that*

$$\|e_1\| \to 0 \quad as \quad t \to \infty$$

provided the control gain k_s introduced in (9) is selected sufficiently large based on the initial conditions of the states, and the remaining control gains are selected based on the following sufficient conditions

$$\alpha_1 > \frac{1}{2}, \qquad \alpha_2 > \beta_2 + \frac{1}{2}, \qquad \beta_2 > \zeta_5,$$

$$\beta > \zeta_1 + \zeta_2 + \frac{1}{\alpha_2}\zeta_3 + \frac{1}{\alpha_2}\zeta_4, \qquad (1-\varphi_2)\gamma > 1 \tag{22}$$

where $\alpha_1, \alpha_2, \beta, \gamma$ were introduced in (2), (3), (10) and (21), φ_2 was introduced in Assumption 1 and β_2 is a subsequently defined gain constant.

Proof: Let $\mathscr{D} \subset \mathbb{R}^{3n+3}$ be a domain containing $y(e_1, e_2, r, P, Q, R_{LK}) \in \mathbb{R}^{3n+3}$, defined as

$$y \triangleq \begin{bmatrix} z & \sqrt{P} & \sqrt{Q} & \sqrt{R_{LK}} \end{bmatrix}. \tag{23}$$

In (23), the auxiliary function $P(e_2, t) \in \mathbb{R}$ is defined as the generalized Filippov solution to the following differential equation

$$\dot{P} \triangleq -r^T (N_{B_1} + N_D - \beta sgn(e_2)) - \dot{e}_2^T N_{B_2} + \beta_2 \|e_2\|^2,$$

$$P(e_2(t_0), t_0) \triangleq \beta \sum_{i=1}^{n} |e_{2i}(t_0)| - e_2(t_0)^T N_D(t_0) \tag{24}$$

where the subscript $i = 1, 2, ..., n$ denotes the ith element of the vector. Similar to the development in (10), existence of solutions for $P(e_2, t)$ can be established using Filippov's theory of differential inclusions for $\dot{P} \in K[h_2](r, \dot{e}_2, e_2, t)$, where $h_2(r, \dot{e}_2, e_2, t) \in \mathbb{R}$ is defined as the right-hand side of $\dot{P}$. Provided the sufficient conditions in (22) are satisfied, $P(e_2, t) \geq 0$ (See [26] for proof). Additionally, the auxiliary function $Q\left(\tilde{W}, \tilde{V}, t\right) \in \mathbb{R}$ in (23) is defined as

$$Q \triangleq \frac{\alpha_2}{2} tr\left(\tilde{W}^T \Gamma_1^{-1} \tilde{W}\right) + \frac{\alpha_2}{2} tr\left(\tilde{V}^T \Gamma_2^{-1} \tilde{V}\right) \tag{25}$$

where $Q \geq 0$ since Γ_1 and Γ_2 are constant, symmetric, and positive definite matrices and $\alpha_2 \in \mathbb{R}^+$.

Let $V(y, t) : \mathscr{D} \times [0, \infty) \to \mathbb{R}$ be a positive-definite, Lipschitz continuous, regular function defined as

$$V \triangleq \frac{1}{2} e_1^T e_1 + \frac{1}{2} e_2^T e_2 + \frac{1}{2} r^T r + P + Q + R_{LK} \tag{26}$$

which satisfies the following inequalities

$$\phi_1(y) \leq V(t) \leq \phi_2(y) \tag{27}$$

where the continuous positive-definite functions $\phi_1(y), \phi_2(y) \in \mathbb{R}$ are defined as $\phi_1(y) \triangleq \lambda_1 \|y\|^2$, $\phi_2(y) \triangleq \lambda_2 \|y\|^2$ and $\lambda_1, \lambda_2 \in \mathbb{R}^+$ are known constants. Under Filippov's framework, a generalized Lyapunov stability theory can be used to establish strong stability of the closed-loop system $\dot{y} = h_3(y, t)$, where $h_3(y, t) \in \mathbb{R}^{3n+3}$ denotes the right-hand side of the closed-loop error signals. The time derivative of (26) exists almost everywhere (a.e.), i.e., for almost all $t \in [t_0, t_f]$, and $\dot{V}(y, t) \overset{a.e.}{\in} \dot{\tilde{V}}(y, t)$ where

$$\dot{\tilde{V}} = \bigcap_{\xi \in \partial V(y,t)} \xi^T K[\varrho]$$

where $\varrho \in \mathbb{R}^{3n+4}$ is defined as $\varrho \triangleq \begin{bmatrix} \dot{e}_1^T & \dot{e}_2^T & \dot{r}^T & \frac{1}{2} P^{-\frac{1}{2}} \dot{P} & \frac{1}{2} Q^{-\frac{1}{2}} \dot{Q} & \frac{1}{2} R_{LK}^{-\frac{1}{2}} \dot{R}_{LK} & 1 \end{bmatrix}^T$, and ∂V is the generalized gradient of $V(y, t)$ [40]. Since $V(y, t)$ is a Lipschitz continuous regular function,

$$\dot{\tilde{V}} \subset \nabla V K[\cdot]^T \tag{28}$$

where

$$\nabla V \triangleq \begin{bmatrix} e_1^T & e_2^T & r^T & 2P^{\frac{1}{2}} & 2Q^{\frac{1}{2}} & 2R_{LK}^{\frac{1}{2}} \end{bmatrix}.$$

Using the calculus for $K[\cdot]$ from [38], and substituting (2)-(4), and (15), (24), the time derivatives of (21), and (25) into (28), yields

$$\begin{aligned} \dot{\tilde{V}} \subset\ & e_1^T (e_2 - \alpha_1 e_1) + e_2^T (r - \alpha_2 e_2) \\ & + r^T \left(\tilde{N} + N_D + N_{B_1} + N_{B_2} + e_2 - (k_s + 1) r\right) \\ & + r^T (-\beta K[sgn(e_2)]) + \beta_2 \|e_2\|^2 \\ & - r^T (N_{B_1} + N_D - \beta K[sgn(e_2)]) - \dot{e}_2^T N_{B_2} \\ & + \frac{\gamma}{2k_s} \rho_2^2(\|z\|) \|z\|^2 - \frac{\gamma(1 - \dot{\tau})}{2k_s} \rho_2^2(\|z_\tau\|) \|z_\tau\|^2 \\ & + tr\left(\alpha_2 \tilde{W}^T \Gamma_1^{-1} \dot{\tilde{W}}\right) + tr\left(\alpha_2 \tilde{V}^T \Gamma_2^{-1} \dot{\tilde{V}}\right) \end{aligned} \tag{29}$$

where $K[sgn(e_2)] = SGN(e_2)$ [38] such that $SGN(e_{2_i}) = 1$ if $e_{2_i}(\cdot) > 0$, $[-1, 1]$ if $e_{2_i}(\cdot) = 0$, and -1 if $e_{2_i}(\cdot) < 0$. Canceling terms and utilizing the bounds from (20) and Assumption 1, we can upper bound (29) as

$$\begin{aligned} \dot{V} \overset{a.e.}{\leq}\ & \|e_1\| \|e_2\| - \alpha_1 \|e_1\|^2 - \alpha_2 \|e_2\|^2 \\ & + \|r\| \rho_1(\|z\|) \|z\| + \|r\| \rho_2(\|z_\tau\|) \|z_\tau\| \\ & - (k_s + 1) \|r\|^2 + \beta_2 \|e_2\|^2 + \frac{\gamma}{2k_s} \rho_2^2(\|z\|) \|z\|^2 \\ & - \frac{\gamma(1 - \varphi_2)}{2k_s} \rho_2^2(\|z_\tau\|) \|z_\tau\|^2 \end{aligned} \tag{30}$$

where the set in (29) reduces to the scalar inequality in (30) since the RHS is continuous a.e., i.e, the RHS is continuous except for the Lebesgue negligible set of times when $e_2(e_1, \dot{e}_1, t) = 0$ [37], [39]. Young's inequality can be used to show that $\|e_1\| \|e_2\| \leq \frac{1}{2} \|e_1\|^2 + \frac{1}{2} \|e_2\|^2$ and $\|r\| \rho_2(\|z_\tau\|) \|z_\tau\| \leq \frac{k_s}{2} \|r\|^2 + \frac{1}{2k_s} \rho_2^2(\|z_\tau\|) \|z_\tau\|^2$, which allows for the following upper bound for (30)

$$\begin{aligned} \dot{V} \overset{a.e.}{\leq}\ & \frac{1}{2} \|e_1\|^2 + \frac{1}{2} \|e_2\|^2 - \alpha_1 \|e_1\|^2 - \alpha_2 \|e_2\|^2 \\ & - \frac{k_s}{2} \|r\|^2 - \|r\|^2 + \beta_2 \|e_2\|^2 + \|r\| \rho_1(\|z\|) \|z\| \\ & + \frac{1}{2k_s} \rho_2^2(\|z_\tau\|) \|z_\tau\|^2 + \frac{\gamma}{2k_s} \rho_2^2(\|z\|) \|z\|^2 \\ & - \frac{\gamma(1 - \varphi_2)}{2k_s} \rho_2^2(\|z_\tau\|) \|z_\tau\|. \end{aligned} \tag{31}$$

If $(1 - \varphi_2) \gamma > 1$, and by completing the squares for $r(e_2, \dot{e}_2, t)$, (31) becomes

$$\begin{aligned} \dot{V} \overset{a.e.}{\leq} & -\left(\alpha_1 - \frac{1}{2}\right) \|e_1\|^2 - \left(\alpha_2 - \beta_2 - \frac{1}{2}\right) \|e_2\|^2 - \|r\|^2 \\ & + \frac{1}{2k_s} \rho_1^2(\|z\|) \|z\|^2 + \frac{\gamma}{2k_s} \rho_2^2(\|z\|) \|z\|^2. \end{aligned} \tag{32}$$

Regrouping similar terms, the expression can be upper bounded by

$$\dot{V} \overset{a.e.}{\leq} -\left(\lambda_3 - \frac{\rho^2(\|z\|)}{2k_s}\right) \|z\|^2 \tag{33}$$

where $\rho^2(\|z\|) \triangleq \rho_1^2(\|z\|) + \gamma\rho_2^2(\|z\|)$ and $\lambda_3 \triangleq min\left\{\alpha_1 - \frac{1}{2},\, \alpha_2 - \beta_2 - \frac{1}{2},\, 1\right\}$. The bounding function $\rho(\|z\|) : \mathbb{R} \to \mathbb{R}$ is a positive-definite, globally invertible function. The expression in (33) can be further upper bounded by a continuous, positive semi-definite function

$$\dot{V} \overset{a.e.}{\leq} -\phi_3(y) = -c\|z\|^2 \qquad \forall y \in \mathscr{D} \tag{34}$$

for some positive constant $c \in \mathbb{R}^+$ and domain $\mathscr{D} = \left\{y \in \mathbb{R}^{3n+3} \mid \|y\| < \rho^{-1}\left(\sqrt{2\lambda_3 k_s}\right)\right\}$. Larger values of k_s will expand the size of the domain $\mathscr{D}$. The inequalities in (27) and (34) can be used to show that $V \in \mathcal{L}_\infty$ in $\mathscr{D}$. Thus, $e_1(\cdot), e_2(\cdot), r(\cdot) \in \mathcal{L}_\infty$ in $\mathscr{D}$. The closed-loop error system can be used to conclude that the remaining signals are bounded in $\mathscr{D}$, and the definitions for $\phi_1(\cdot)$ and $z(\cdot)$ can be used to show that $\phi_1(\cdot)$ is uniformly continuous in $\mathscr{D}$. Let $S_{\mathscr{D}} \subset \mathscr{D}$ denote a set defined as

$$\mathcal{S}_{\mathscr{D}} \triangleq \left\{y \in \mathscr{D} \mid \phi_2 < \lambda_1\left(\rho^{-1}\left(\sqrt{2\lambda_3 k_s}\right)\right)^2\right\}. \tag{35}$$

The region of attraction in (35) can be made arbitrarily large to include any initial conditions by increasing the control gain k_s. From (34), [41, Corollary 1] can be invoked to show that $c\|z\|^2 \to 0 \;\; as \;\; t \to \infty \; \forall y(0) \in \mathcal{S}_{\mathscr{D}}$. Based on the definition of $z(\cdot)$ in (20), $\|e_1\| \to 0 \; as \; t \to \infty \, \forall y(0) \in \mathcal{S}_{\mathscr{D}}$. ■

V. Conclusion

A continuous, neural network augmented, RISE controller is utilized for uncertain nonlinear systems which include unknown, arbitrarily large, time-varying state delays and additive bounded disturbances. The controller assumes the time-delay is bounded and slowly varying. Time-varying LK functionals are utilized to prove semi-global asymptotic tracking of the closed-loop system in the presence of time-varying and non-LP functions and sufficiently smooth unmodeled dynamic effects. Future goals include investigating methods to eliminate the assumption on the rate of the change of the delay and experimental demonstration of the developed controller.

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