LaSalle-Yoshizawa Corollaries for Nonsmooth Systems

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***Abstract*—In this note, two generalized corollaries to the LaSalle-Yoshizawa Theorem are presented for nonautonomous systems described by nonlinear differential equations with discontinuous right-hand sides. Lyapunov-based analysis methods that achieve asymptotic convergence when the candidate Lyapunov derivative is upper bounded by a negative semi-definite function in the presence of differential inclusions are presented. A design example illustrates the utility of the corollaries.**

I. Introduction

Various Lyapunov-based analysis methods have been developed for differential inclusions in literature for both autonomous (cf. [1]–[9]) and nonautonomous (cf. [6], [10]–[13]) systems. Of these, several stability theorems have been established which apply to nonsmooth systems for which the derivative of the candidate Lyapunov function can be upper bounded by a negative-definite function: Lyapunov's generalized theorem and finite-time convergence in [8], [10]–[14] are examples of such. However, for certain classes of controllers (e.g., adaptive controllers, output feedback controllers, etc.), a negative-definite bound may be difficult (or impossible) to achieve, restricting the use of such methods.

Matrosov's Theorem [15] provides a framework for examining the stability of equilibrium points (and sets through various extensions) when the candidate Lyapunov function has a negative semi-definite decay. Various extensions of this theorem have been developed (cf. [16]–[20]) to encompass discrete and hybrid systems to establish stability of closed sets. In particular, [19] (see also the related work in [16] and [17]) extended Matrosov's Theorem to differential inclusions, while also addressing the stability of sets.

In contrast to Matrosov Theorems, LaSalle's Invariance Principle [21] has been widely adopted as a method, for continuous autonomous (time-invariant) systems, to relax the strict negative-definite condition on the candidate Lyapunov function derivative while still ensuring asymptotic stability of the origin. Stability of the origin is proven by showing that bounded solutions converge to the largest invariant subset contained in the set of points where the derivative of the candidate Lyapunov function is zero. In [22], LaSalle's Invariance Principle was modified to state that bounded solutions converge to the largest invariant subset of the set where an integrable output function is zero. The integral invariance method was further extended in [23] to differential inclusions. As described in [24], additional extensions of the invariance principle to systems with discontinuous right-hand sides were presented in [4], [6], [9] for Filippov solutions and [25] for Carathéodory solutions.

Various extensions of LaSalle's Invariance Principle have also been developed for hybrid systems (cf. [24], [26]–[30]). The results in [26] and [29] focus on switched linear systems, whereas the result in [30] focuses on switched nonlinear systems. In [28], hybrid extensions of LaSalle's Invariance Principle were applied for systems where at least one solution exists for each initial condition for deterministic systems and continuous hybrid systems. Left-continuous and impulsive hybrid systems are considered in extensions in [27]. In [24], two invariance principles are developed for hybrid systems: one involves a Lyapunov-like function that is nonincreasing along all trajectories that remain in a given set, and the other considers a pair of auxiliary output functions that satisfy certain conditions only along the hybrid trajectory. A review of invariance principles for hybrid systems is provided in [31].

The challenge for developing invariance-like principles for nonautonomous systems is that it may be unclear how to even define a set where the derivative of the candidate Lyapunov function is stationary since the candidate Lyapunov function is a function of both state and time [32], [33]. By augmenting the state vector with time (cf. [34], [35]), a nonautonomous system can be expressed as an autonomous system: this technique allows autonomous systems results (cf. [36] and [37]) to be extended to nonautonomous systems. While the state augmentation method can be a useful tool, in general, augmenting the state vector yields a non-compact attractor (when the time dependence is not periodic), destroying some of the structure of the original equation; for example, the new system will not have any bounded, periodic, or almost periodic motions. Some results (cf. [38]–[40]) have explored ways to utilize the augmented system's non-compact attractors by focusing on solution operator decomposition, energy equations or new notions of compactness, but these methods typically require additional regularity conditions (with respect to time) than cases when time is kept as a distinct variable.

The Krasovskii-LaSalle Theorem [41] was originally developed for periodic systems, with several generalizations also existing for not necessarily periodic systems (e.g., see [6], [42]–[45]). In particular, a (Krasovskii-LaSalle) Extended Invariance Principle is developed in [45] to prove that the origin of a nonautonomous switched system with a piecewise continuous uniformly bounded in time right-hand side is globally asymptotically stable (or uniformly globally asymptotically stable for autonomous systems). The result in [45] uses a Lipschitz continuous, radially unbounded, positive-definite function with a negative semi-definite derivative (condition C1) along with an auxiliary Lipschitz continuous (possibly indefinite) function whose derivative is upper bounded by terms whose sum are positive-definite (condition C2).

Also for nonautonomous systems, the LaSalle-Yoshizawa Theorem (i.e., [33, Theorem 8.4] and [46, Theorem A.8]), based on the work in [21], [47], [48], provides a convenient analysis tool which allows the limiting set (which does not need to be invariant) to be defined where the negative semi-definite bound on the candidate Lyapunov derivative is equal to zero, guaranteeing asymptotic convergence of the state. Given its utility, the LaSalle-Yoshizawa Theorem has been applied, for example, in adaptive control and in deriving stability from passivity properties such as feedback passivation and backstepping designs of nonlinear systems [21]. Available proofs for the LaSalle-Yoshizawa Theorem exploit Barbalat's Lemma, which is often invoked to show asymptotic convergence for general classes of nonlinear systems [33]. In general, adapting the LaSalle-Yoshizawa Theorem to systems where

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the right-hand side is not locally Lipschitz has only recently been explored. The result in [49] presents three invariance-like semistability theorems that utilize similar arguments to the LaSalle-Yoshizawa Theorem under the assumption that the system dynamics are uniformly bounded. Alternatively, using Barbalat's Lemma and the observation that an absolutely continuous function that has a uniformly locally integrable derivative is uniformly continuous, the result in [50] proves asymptotic convergence of an output function for nonlinear systems with L_p disturbances. The result in [50] is developed for differential equations with a continuous right-hand side, but [50, Facts 1-4] provide insights into the application of Barbalat's Lemma to discontinuous systems.

In this paper, we present two corollaries to the LaSalle-Yoshizawa Theorem for nonautonomous systems with right-hand side discontinuities that are essentially locally bounded, uniformly in t, utilizing Filippov solutions and Lipschitz continuous and regular Lyapunov-like functions whose time derivatives can be upper bounded by negative semi-definite functions. Applicability of one of the corollaries is illustrated for an example problem.

II. Preliminaries

Consider the system

$$\dot{x} = f(x,t) \tag{1}$$

where $x(t) \in \mathcal{D} \subset \mathbb{R}^n$ denotes the state vector, $f : \mathcal{D} \times [0,\infty) \to \mathbb{R}^n$ is Lebesgue measurable and essentially locally bounded, uniformly in t and $\mathcal{D}$ is an open and connected set. Existence and uniqueness of the continuous solution $x(t)$ are provided under the condition that the function f is Lipschitz continuous. However, if f contains a discontinuity at any point in $\mathcal{D}$, then a solution to (1) may not exist in the classical sense. Thus, it is necessary to redefine the concept of a solution. Utilizing differential inclusions, the value of a generalized solution (e.g., Filippov [51] or Krasovskii [52] solutions) at a certain point can be found by interpreting the behavior of its derivative at nearby points. Generalized solutions will be close to the trajectories of the actual system since they are a limit of solutions of ordinary differential equations with a continuous right-hand side [13]. While there exists a Filippov solution for any arbitrary initial condition $x(t_0) \in \mathcal{D}$, the solution is generally not unique [5], [51].

Definition 1. (Filippov Solution) [51] A function $x : [0,\infty) \to \mathbb{R}^n$ is called a solution of (1) on the interval $[0,\infty)$ if $x(t)$ is absolutely continuous and for almost all $t \in [0,\infty)$,

$$\dot{x} \in K[f](x(t),t)$$

where $K[f](x(t),t)$ is an upper semi-continuous, nonempty, compact and convex valued map on $\mathcal{D}$, defined as

$$K[f](x(t),t) \triangleq \bigcap_{\delta>0} \bigcap_{\mu N=0} \overline{co} f(B(x(t),\delta)\setminus N, t), \tag{2}$$

$\bigcap_{\mu N=0}$ denotes the intersection over sets N of Lebesgue measure zero, $\overline{co}$ denotes convex closure, and $B(x(t),\delta) = \{v \in \mathbb{R}^n | \ \|x(t) - v\| < \delta\}$.

Remark 1. One can also formulate the solutions of (1) in other ways [53]; for instance, using Krasovskii's definition of solutions [52]. The corollaries presented in this work can also be extended to Krasovskii solutions (see [3], for example). In the case of Krasovskii solutions, one would get stronger conclusions (i.e., conclusions for a potentially larger set of solutions) at the cost of slightly stronger assumptions (e.g., local boundedness rather than essentially local boundedness).

To facilitate the main results, four definitions are provided.

Definition 2. (Directional Derivative) [54] Given a function $f : \mathbb{R}^m \to \mathbb{R}^n$, the right directional derivative of f at $x \in \mathbb{R}^m$ in the direction of $v \in \mathbb{R}^m$ is defined as

$$f'(x,v) = \lim_{t\to 0^+} \frac{f(x+tv) - f(x)}{t}.$$

Additionally, the generalized directional derivative of f at x in the direction of v is defined as

$$f^o(x,v) = \lim_{y\to x} \sup_{t\to 0^+} \frac{f(y+tv) - f(y)}{t}.$$

Definition 3. (Regular Function) [34] A function $f : \mathbb{R}^m \to \mathbb{R}^n$ is said to be regular at $x \in \mathbb{R}^m$ if for all $v \in \mathbb{R}^m$, the right directional derivative of f at x in the direction of v exists and $f'(x,v) = f^o(x,v)$.[1]

Definition 4. (Clarke's Generalized Gradient) [34] For a function $V : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ that is locally Lipschitz in (x,t), define the generalized gradient of V at (x,t) by

$$\partial V(x,t) = \overline{co}\{\lim \nabla V(x,t) \mid (x_i,t_i) \to (x,t), (x_i,t_i) \notin \Omega_V\}$$

where Ω_V is the set of measure zero where the gradient of V is not defined.

Definition 5. (Locally bounded, uniformly in t) Let $f : \mathcal{D} \times [0,\infty) \to \mathbb{R}$. The map $x \to f(x,t)$ is locally bounded, uniformly in t, if for each compact set $K \subset \mathcal{D}$, there exists $c > 0$ such that $|f(x,t)| \leq c$, $\forall (x,t) \in K \times [0,\infty)$.

The following lemma provides a method for computing the time derivative of a regular function V using Clarke's generalized gradient [34] and $K[f](x,t)$, from (2), along the solution trajectories of (1).

Lemma 1. (Chain Rule) *[6], [56] Let $x(t)$ be a Filippov solution of (1) and $V : \mathcal{D} \times [0,\infty) \to \mathbb{R}$ be a locally Lipschitz, regular function. Then $V(x(t),t)$ is absolutely continuous, $\frac{d}{dt}V(x(t),t)$ exists almost everywhere (a.e.), i.e., for almost all $t \in [0,\infty)$, and $\dot{V}(x(t),t) \overset{a.e.}{\in} \dot{\tilde{V}}(x(t),t)$, where*[2]

$$\dot{\tilde{V}}(x,t) \triangleq \bigcap_{\xi\in\partial V(x,t)} \xi^T \begin{pmatrix} K[f](x,t) \\ 1 \end{pmatrix}.$$

Remark 2. Throughout the subsequent discussion, for brevity of notation, let a.e. refer to almost all $t \in [0,\infty)$.

III. Main Result

For the system described in (1) with a continuous right-hand side, existing Lyapunov theory can be used to examine the stability of the closed-loop system using continuous techniques such as those described in [57]. However, these theorems must be altered for the set-valued map $\dot{\tilde{V}}(x(t),t)$ for systems with right-hand sides which are not Lipschitz continuous [6], [13], [14]. Lyapunov analysis for nonsmooth systems is analogous to the analysis used for continuous systems. The differences are that differential equations are replaced with inclusions, gradients are replaced with generalized gradients, and points are replaced with sets throughout the analysis. The following presentation and subsequent proofs demonstrate how the LaSalle-Yoshizawa Theorem can be adapted for such systems.

[1] Note that any $\mathcal{C}^1$ continuous function is regular and the sum of regular functions is regular [55].

[2] Equivalently, almost everywhere, $\dot{V} = \xi^T \begin{pmatrix} \eta \\ 1 \end{pmatrix}$ for all $\xi \in \partial V(x,t)$ and some $\eta \in K[f](x,t)$.

The following auxiliary lemma from [56] and Barbalat's Lemma are provided to facilitate the proofs of the nonsmooth LaSalle-Yoshizawa Corollaries.

Lemma 2. *[56] Let $x(t)$ be any Filippov solution to the system in (1) and $V : \mathcal{D}\times[0,\infty) \to \mathbb{R}$ be a locally Lipschitz, regular function. If $\dot{V}(x(t),t) \overset{a.e.}{\leq} 0$, then $V(x(t),t) \leq V(x(t_0),t_0)$ $\forall t > t_0$.*

Proof: For the sake of contradiction, let there exist some $t > t_0$ such that $V(x(t),t) > V(x(t),t_0)$. Then,

$$\int_{t_0}^{t} \dot{V}(x(\sigma),\sigma)\, d\sigma = V(x(t),t) - V(x(t),t_0) > 0.$$

It follows that $\dot{V}(x(t),t) > 0$ on a set of positive measure, which contradicts that $\dot{V}(x(t),t) \leq 0$, a.e. ∎

Lemma 3. (Barbalat's Lemma) *[57] Let $\phi : \mathbb{R}^+ \to \mathbb{R}$ be a uniformly continuous function. Suppose that $\lim_{t\to\infty} \int_0^t \phi(\tau)\, d\tau$ exists and is finite. Then,*

$$\phi(t) \to 0 \quad as \quad t \to \infty.$$

Based on Lemmas 2 and 3, nonsmooth corollaries to the LaSalle-Yoshizawa Theorem (c.f., [33, Theorem 8.4] and [46, Theorem A.8]) are provided in Corollary 1 and 2.

Corollary 1. *For the system given in (1), let $\mathcal{D} \subset \mathbb{R}^n$ be an open and connected set containing $x = 0$ and suppose f is Lebesgue measurable and $x \to f(x,t)$ is essentially locally bounded, uniformly in t. Let $V : \mathcal{D} \times [0,\infty) \to \mathbb{R}$ be locally Lipschitz and regular such that*

$$W_1(x) \leq V(x,t) \leq W_2(x) \quad \forall t \geq 0,\ \forall x \in \mathcal{D}, \tag{3}$$

$$\dot{V}(x(t),t) \overset{a.e.}{\leq} -W(x(t)) \tag{4}$$

where W_1 and W_2 are continuous positive definite functions, W is a continuous positive semi-definite function on $\mathcal{D}$, choose $r > 0$ and $c > 0$ such that $B_r \subset \mathcal{D}$ and $c < \min_{\|x\|=r} W_1(x)$ and $x(t)$ is a Filippov solution to (1) where $x(t_0) \in \{x \in B_r \mid W_2(x) \leq c\}$. Then $x(t)$ is bounded and satisfies

$$W(x(t)) \to 0 \quad as \quad t \to \infty. \tag{5}$$

Proof: Since $B_r \subset \mathcal{D}$ and $c < \min_{\|x\|=r} W_1(x)$, $\{x \in B_r \mid W_1(x) \leq c\}$ is in the interior of B_r. Define a time-dependent set $\Omega_{t,c}$ by

$$\Omega_{t,c} = \{x \in B_r \mid V(x,t) \leq c\}.$$

From (3), the set $\Omega_{t,c}$ contains $\{x \in B_r \mid W_2(x) \leq c\}$ since

$$W_2(x) \leq c \Rightarrow V(x,t) \leq c.$$

On the other hand, $\Omega_{t,c}$ is a subset of $\{x \in B_r \mid W_1(x) \leq c\}$ since

$$V(x,t) \leq c \Rightarrow W_1(x) \leq c.$$

Thus,

$$\{x \in B_r \mid W_2(x) \leq c\} \subset \Omega_{t,c} \subset \{x \in B_r \mid W_1(x) \leq c\} \subset B_r \subset \mathcal{D}.$$

Based on (4), $\dot{V}(x(t),t) \overset{a.e.}{\leq} 0$, hence, $V(x(t),t)$ is non-increasing from Lemma 2. For any $t_0 \geq 0$ and any $x(t_0) \in \Omega_{t_0,c}$, the solution starting at $(x(t_0),t_0)$ stays in $\Omega_{t,c}$ for every $t \geq t_0$. Therefore, any solution starting in $\{x \in B_r \mid W_2(x) \leq c\}$ stays in $\Omega_{t,c}$, and consequently in $\{x \in B_r \mid W_1(x) \leq c\}$, for all future time. Hence, the Filippov solution $x(t)$ is bounded such that $\|x(t)\| < r$, $\forall t \geq t_0$.

From Lemma 2, $V(x(t),t)$ is also bounded such that $V(x(t),t) \leq V(x(t_0),t_0)$. Since $\dot{V}(x(t),t)$ is Lebesgue measurable from (4),

$$\int_{t_0}^{t} W(x(\tau))\, d\tau \leq -\int_{t_0}^{t} \dot{V}(x(\tau),\tau)\, d\tau = V(x(t_0),t_0) - V(x(t),t) \leq V(x(t_0),t_0). \tag{6}$$

Therefore, $\int_{t_0}^{t} W(x(\tau))\, d\tau$ is bounded $\forall t > t_0$. Existence of $\lim_{t\to\infty} \int_{t_0}^{t} W(x(\tau))\, d\tau$ is guaranteed since the left-hand side of (6) is monotonically nondecreasing (based on the definition of $W(x)$) and bounded above. Since $x(t)$ is locally absolutely continuous and f is essentially locally bounded, uniformly in t, $x(t)$ is uniformly continuous. Because $W(x)$ is continuous in x, and x is on the compact set B_r, $W(x(t))$ is uniformly continuous in t on $(t_0,\infty]$. Therefore, by Lemma 3, $W(x(t)) \to 0\ as\ t \to \infty$. ∎

Remark 3. From Def. 1, $K[f](x,t)$ is an upper semi-continuous, nonempty, compact and convex valued map. While existence of a Filippov solution for any arbitrary initial condition $x(t_0) \in \mathcal{D}$ is provided by the definition, generally speaking, the solution is non-unique [5], [51].

Note that Corollary 1 establishes (5) for a specific $x(t)$. Under the stronger condition that[3] $\dot{\tilde{V}}(x,t) \leq W(x)\, \forall x \in \mathcal{D}$, it is possible to show that (5) holds for all Filippov solutions of (1). The next corollary is presented to illustrate this point.

Corollary 2. *For the system given in (1), let $\mathcal{D} \subset \mathbb{R}^n$ be an open and connected set containing $x = 0$ and suppose f is Lebesgue measurable and $x \to f(x,t)$ is essentially locally bounded, uniformly in t. Let $V : \mathcal{D} \times [0,\infty) \to \mathbb{R}$ be locally Lipschitz and regular such that*

$$W_1(x) \leq V(x,t) \leq W_2(x) \tag{7}$$

$$\dot{\tilde{V}}(x,t) \leq -W(x) \tag{8}$$

$\forall t \geq 0$, $\forall x \in \mathcal{D}$ where W_1 and W_2 are continuous positive definite functions, and W is a continuous positive semi-definite function on $\mathcal{D}$. Choose $r > 0$ and $c > 0$ such that $B_r \subset \mathcal{D}$ and $c < \min_{\|x\|=r} W_1(x)$. Then, all Filippov solutions of (1) such that $x(t_0) \in \{x \in B_r \mid W_2(x) \leq c\}$ are bounded and satisfy

$$W(x(t)) \to 0\ \ as\ \ t \to \infty.$$

Proof: Let $x(t)$ be any arbitrary Filippov solution of (1). Then, from Lemma 1, and (8), $\dot{V}(x(t),t) \overset{a.e.}{\leq} -W(x(t))$, which is the condition (4). Since the selection of $x(t)$ is arbitrary, Corollary 1 can be used to imply that the result in (5) holds for each $x(t)$. ∎

IV. Design Example

The LaSalle-Yoshizawa Corollaries (and the LaSalle-Yoshizawa Theorem) are useful in their ability to provide boundedness and convergence of solutions, while providing a compact framework to define the region of attraction for which boundedness and convergence results hold. In fact, the region of attraction is provided as part of the corollary structures. In the case of semi-global and local results, these domains and sets are especially useful. It is important to note that Barbalat's Lemma can be used to achieve the same results (in fact, it is used in the proof for Corollary 1); however, the use of Barbalat's Lemma would require the identification of the region of attraction for which convergence holds and does not provide boundedness of the trajectories. For illustrative purposes,

[3]The inequality $\dot{\tilde{V}}(x,t) \leq W(x)$ is used to indicate that every element of the set $\dot{\tilde{V}}(x,t)$ is less than or equal to the scalar $W(x)$.

the following design example targets the regulation of a first order nonlinear system.

Consider a first order nonlinear differential equation given by

$$\dot{x} = f(x,t) + d(x,t) + u(t) \quad (9)$$

where $f : \mathbb{R}^n \times [0,\infty) \rightarrow \mathbb{R}^n$ is an unknown, linear-parameterizable, essentially locally bounded, uniformly in t function that can be expressed as $f(x,t) = Y(x,t)\,\theta$ where $\theta \in \mathbb{R}^p$ is a vector of unknown constant parameters and $Y : \mathbb{R}^n \times [0,\infty) \rightarrow \mathbb{R}^{n\times p} \times [0,\infty)$ is the regression matrix for $f(x,t)$. In addition, $u : [0,\infty) \rightarrow \mathbb{R}^n$ is the control input, $x(t) \in \mathbb{R}^n$ is the measurable system state, and $d : \mathbb{R}^n \times [0,\infty) \rightarrow \mathbb{R}^n$ is an essentially locally bounded disturbance that satisfies

$$\|d(x,t)\| \leq c_1 + c_2(\|x\|)\,\|x\| \quad (10)$$

where $c_1 \in \mathbb{R}^+$ is a positive constant, and $c_2 : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a positive, globally invertible, state-dependent function. A regulation controller for (9) can be designed as

$$u(x,t) \triangleq -k_1 x - k_2 sgn(x) - Y\hat{\theta} \quad (11)$$

where $\hat{\theta}(x,t) \in \mathbb{R}^p$ is an estimate of θ, $k_1, k_2 \in \mathbb{R}^+$ are gain constants, and $sgn(\cdot)$ is defined $\forall \xi \in \mathbb{R}^n = \begin{bmatrix} \xi_1 & \xi_2 & \ldots & \xi_n \end{bmatrix}^T$ as $sgn(\xi) \triangleq \begin{bmatrix} sgn(\xi_1) & sgn(\xi_2) & \ldots & sgn(\xi_n) \end{bmatrix}^T$. Based on the subsequent stability analysis, an adaptive update law can be defined as

$$\dot{\hat{\theta}} \triangleq \Gamma Y^T x \quad (12)$$

where $\Gamma \in \mathbb{R}^{n\times n}$ is a positive gain matrix. The closed-loop system is given by

$$\dot{x} = Y\tilde{\theta} + d(x,t) - k_1 x - k_2 sgn(x) \quad (13)$$

where $\tilde{\theta}(t) \in \mathbb{R}^p$ denotes the mismatch $\tilde{\theta} \triangleq \theta(t) - \hat{\theta}(t)$. In (13), it is apparent that the RHS contains a discontinuity in $x(t)$, and the use of differential inclusions provides for existence of solutions. Let $y\left(x,\tilde{\theta}\right) \in \mathbb{R}^{n+p}$ be defined as $y \triangleq \begin{bmatrix} x^T & \tilde{\theta}^T \end{bmatrix}^T$ and choose a positive-definite, locally Lipschitz, regular candidate Lyapunov function as

$$V(y,t) \triangleq \frac{1}{2}x^T x + \frac{1}{2}\tilde{\theta}^T \Gamma^{-1}\tilde{\theta}. \quad (14)$$

The candidate Lyapunov function in (14) satisfies the following inequalities:

$$W_1(y) \leq V(y,t) \leq W_2(y) \quad (15)$$

where the continuous positive-definite functions $W_1, W_2 : \mathbb{R}^{n+p} \rightarrow \mathbb{R}^+$ are defined as $W_1(y) \triangleq \lambda_1 \|y\|^2$, and $W_2(y) \triangleq \lambda_2 \|y\|^2$, where $\lambda_1, \lambda_2 \in \mathbb{R}^+$ are known constants. Then, $\dot{V}(y(t),t) \overset{a.e.}{\in} \dot{\tilde{V}}(y(t),t)$ and

$$\dot{\tilde{V}} \triangleq \bigcap_{\xi \in \partial V\left(x,\tilde{\theta},t\right)} \xi^T K \begin{bmatrix} \dot{x} \\ \dot{\tilde{\theta}} \\ 1 \end{bmatrix} \left(x,\tilde{\theta},t\right).$$

Since $V(y,t)$ is $\mathcal{C}^\infty$ in y,[4]

$$\dot{\tilde{V}} \subset \nabla V^T K \begin{bmatrix} \dot{x} \\ \dot{\tilde{\theta}} \end{bmatrix} \left(x,\tilde{\theta}\right) \subset \left[x^T, \tilde{\theta}^T \Gamma^{-1}\right] K \begin{bmatrix} \dot{x} \\ \dot{\tilde{\theta}} \end{bmatrix} \left(x,\tilde{\theta}\right). \quad (16)$$

After using (13), the expression in (16) can be written as

$$\dot{\tilde{V}} \subset x^T \Big(Y\tilde{\theta} + d(x,t) - k_1 x - k_2 K[sgn(x)] \Big) - \tilde{\theta}^T \Gamma^{-1} \dot{\hat{\theta}} \quad (17)$$

[4]For continuously differentiable Lyapunov candidate functions, the generalized gradient reduces to the standard gradient. However, this is not required by the Corollary itself and only assists in evaluation.

where $K[sgn(x)] = SGN(x)$ such that $SGN(x_i) = 1$ if $x_i > 0$, $[-1,1]$ if $x_i = 0$, and -1 if $x_i < 0$ $\forall i = 1,2,...,n$.

Remark 4. One could also consider the discontinuous function instead of the differential inclusion (i.e., the $sgn(\cdot)$ function can alternatively be defined as $sgn(0) = 0$) using Caratheodory solutions; however, this method would not be an indicator for what happens when measurement noise is present in the system. As described in results such as [58]–[60], Filippov and Krasovskii solutions for discontinuous differential equations are appropriate for capturing the possible closed-loop system behavior in the presence of arbitrarily small measurement noise. By utilizing the set valued map $SGN(\cdot)$ in the analysis, we account for the possibility that when the true state satisfies $x = 0$, $sgn(x)$ (of the measured state) falls within the set $[-1,1]$. Therefore, the presented analysis is more robust to measurement noise than an analysis that depends on $sgn(0)$ to be defined as a known singleton.

Substituting for the adaptive update law in (12), canceling terms and utilizing the bound for d in (10), the expression in (16) can be upper bounded as

$$\dot{\tilde{V}} \leq -k_1 \|x\|^2 + c_1 \|x\| + c_2(\|x\|)\,\|x\|^2 - k_2 \|x\|. \quad (18)$$

The set in (17) reduces to the scalar inequality in (18) since in the case when $K[sgn(x)]$ is defined as a set, it is multiplied by x, i.e., when $x = 0$, $0 \cdot SGN(0) = \{0\}$. Regrouping similar terms, the expression in (18) can be written as

$$\dot{\tilde{V}} \leq -(k_1 - c_2(\|x\|))\,\|x\|^2 - (k_2 - c_1)\,\|x\|. \quad (19)$$

Provided $k_2 > c_1$ and $k_1 > c_2(\|x\|)$, the expression in (19) can be upper bounded as $\dot{\tilde{V}}(y,t) \leq -W(y)$ where $W : \mathbb{R}^{n+p} \rightarrow \mathbb{R}^+$ is a positive semi-definite function defined on the domain $\mathcal{D} \triangleq \left\{\sigma \in \mathbb{R}^{n+p} \mid \|\sigma\| < c_2^{-1}(k_1)\right\}$. The inequalities in (15) can be used to show that $V(y(t),t) \in \mathcal{L}_\infty$ in $\mathcal{D}$; hence, $x(t)$ and $\tilde{\theta}(x(t),t) \in \mathcal{L}_\infty$ in $\mathcal{D}$. Since θ contains the constant unknown system parameters and $\tilde{\theta}(x(t),t) \in \mathcal{L}_\infty$ in $\mathcal{D}$, the definition for $\tilde{\theta}(x(t),t)$ can be used to show that $\hat{\theta}(x(t),t) \in \mathcal{L}_\infty$ in $\mathcal{D}$. Given that $x(t) \in \mathcal{L}_\infty$ in $\mathcal{D}$, $Y(x(t),t) \in \mathcal{L}_\infty$ in $\mathcal{D}$. Since $x(t), \hat{\theta}(x(t),t)$, and $Y(x(t),t) \in \mathcal{L}_\infty$ in $\mathcal{D}$, the control is bounded from (11) and the adaption law in (12). The closed-loop dynamics in (10) and (13) can be used to conclude that $\dot{x}(t) \in \mathcal{L}_\infty$ in $\mathcal{D}$; hence, $x(t)$ is uniformly continuous in $\mathcal{D}$.

Choose $0 < r < c_2^{-1}(k_1)$ such that $B_r \subset \mathcal{D}$ denotes a closed ball, and let $\mathcal{S} \subset B_r$ denote the set defined as

$$\mathcal{S} \triangleq \left\{\sigma \subset B_r \mid W_2(\sigma) < \min_{\|\sigma\|=r} W_1(\sigma) = \lambda_1 r^2\right\}. \quad (20)$$

Invoking Corollary 2, $W(y(t)) = -(k_1 - c_2(\|x(t)\|))\,\|x(t)\|^2 \rightarrow 0 \;\; as \;\; t \rightarrow \infty \;\; \forall y(0) \in \mathcal{S}$, thus, $x \rightarrow 0 \;\; as \;\; t \rightarrow \infty \;\; \forall y(0) \in \mathcal{S}$. The region of attraction in (20) can be made arbitrarily large to include all initial conditions (a semi-global type result) by increasing the gain k_1.

Remark 5. For some systems (e.g., closed-loop error systems with sliding mode control laws), it may be possible to show that Corollary 2 is more easily applied, as is the focus of the example in Section IV. However, in other cases, it may be difficult to satisfy the inequality in (8). The usefulness of Corollary 1 is demonstrated in those cases where it is difficult or impossible to show that the inequality in (8) can be satisfied, but it is possible to show that (4) can be satisfied for almost all time.

V. Conclusion

In this paper, the Lasalle-Yoshizawa Theorem is extended to differential systems whose right-hand sides are discontinuous. The result presents two theoretical tools applicable to nonautonomous systems with discontinuities in the closed-loop error system. Generalized Lyapunov-based analysis methods are developed utilizing differential inclusions in the sense of Filippov to achieve asymptotic convergence when the candidate Lyapunov derivative is upper bounded by a negative semi-definite function. Cases when the bound on the Lyapunov derivative holds for all possible Filippov solutions are also considered. An adaptive, sliding mode control example is provided to illustrate the utility of the main results.

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