A METHOD FOR SOLVING SWITCHED OPTIMAL CONTROL PROBLEMS WITH DWELL TIME CONSTRAINTS

By

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A METHOD FOR SOLVING SWITCHED OPTIMAL
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Abstract: Control for dynamical systems with continuous-valued variables have been well explored, but most modern digital control systems are switched, meaning they have continuous- as well as discrete-valued decision variables, and these systems often have constraints on the switching rate called a minimum dwell time constraint. These switched systems can be solved by embedding the switched problem, which means implementing the discrete control variables as continuous, so all variables are continuous-valued and the embedded problem can be solved using conventional techniques. However, the optimal solution to the embedded problem may have control values that cannot be matched by the switched system, requiring further steps to implement and needing additional modifications to take into account constraints on the states or control. This paper introduces a method to ensure that the optimal solutions to the embedded problem are a bang-bang type, which means these solutions can be directly matched to the switched control values. To accomplish this, a cost function is added to the overall cost that penalizes the control signal when the control value cannot be represented by the switched control. The optimal solution will obtain the control values that minimize the cost, thus will be those values that can be attained by the switched control. The weighting on this cost function is directly related to the dwell time for the system, with a higher weight corresponding to a greater dwell time and a slower switching rate and vice versa. To show this, the system is augmented so that the control becomes a new state whose derivative is now the control of the augmented system, and a proof is detailed showing the connection between the weighting of the auxiliary cost on the control and the dwell time of the system. Simulations compare this method to others to show the efficacy of this method.

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CHAPTER I

Introduction

I.1 Background

The field of control feedback touches almost every aspect of life, from cellular respiration in a cell in the human body to the battery control system in a Tesla Model 3 to the industrial process creating fertilizer in a factory. All of these processes have to be controlled by inputs to prevent them from spiraling out of control and to achieve a desired output. To that end, the process is first modeled with a system of dynamical equations that represent how the process changes over time. The states of the system are the set of variables that are evolving over time with respect to the dynamical equations, with the collection of all possible states called the state space. The output is another set of equations of the inputs and the states, which can be as simple as the states themselves. The input to the system is the control, which needs to be designed such that the system achieves the above goals. The process of determining the control such that the system is guided to a desired output and the analysis of these controlled systems is control theory. Optimal control theory is a subset where the system is controlled in such a way that the total cost of the system is minimized, where the cost is a function of the controls and states of the system that outputs a single value. The optimal control is the control signal that has the lowest cost of all other possible control signals, with the associated optimal states of the system. A very good starting point for optimal control theory is [2], which lays out the basics of optimal control theory and various methods for solving optimal control problems (OCPs).

A variable in a system can be either continuous or discrete, where a continuous variable can be any value in a given range between two numbers, while a discrete variable can only be from a finite number of values, such as a binary variable that can only be 0 or 1. A normal OCP has all continuous variables and has generally been solved using some form of the gradient method, which uses the derivative of the system equations and the cost function with respect to the controls to find the lowest possible cost while getting the system to the desired output. These methods can be very fast, since the optimal solution is found directly without having to compute solutions that are not optimal. Years of research has gone into refining algorithms and solving potential problems, such as when a local minimum cost is found rather than the global minimum which is the true optimal cost. However, methods that rely on the gradient technique, which is often used for finding optimal solutions to OCPs, cannot be used on the category of problems where some of the variables are continuous but some are not, which are called hybrid optimal control problems (HOCPs).

This category of OCPs has both continuous and discrete variables, both of which can be control or state variables. As more and more modern systems become digital, HOCPs have come to the forefront of research since the binary nature of digital systems often mean the mathematical models are hybrid, so ways to solve HOCPs are needed. However, these take much more computational power because the discrete variables prevent differentiation which requires all continuous variables, so methods using the gradient cannot be used, thus other methods have to be found. In the worst case, every possible combination of the discrete variables has to be tested to find a solution, and this grows exponentially with an increase in the number of discrete variables, thus HOCPs can take much longer to solve than a normal OCP. Therefore tricks have been found to simplify the problem or exploit characteristics of the problem setup to simplify them and make finding the solution of the problem take less time and less computational power than the brute force method.

A subcategory of HOCPs are switched optimal control problems (SOCPs), in which the discrete variables are only in the control of the system while other controls and the states of the system are continuous. In some cases the discrete control is not a variable in the equations but a selection of which equations govern the system at any one time, which is called the mode of the system. One example of this is in an automatic transmission; the selected gear can be represented by an integer control variable, and each value of the control has a different equation representing the selected gear dynamics from the engine to the output shaft. An important constraint on a switched system that is often present in real world processes is a minimum dwell time constraint, which sets an allowed minimum time between switches in the discrete controller. An example is a refrigeration system, where the number of compressors that are on and the state of the valves are all discrete controls, while the amount of time the valve is on is a continuous control and the states of the system are continuous. The minimum dwell time constraint comes from the fact that the compressors can only be turned on or off after a certain amount of time has passed since the last time it switched because of physical characteristics of the compressor. In the following paper, a new method for solving

SOCPs will be introduced that can efficiently solve these problems while also being able to directly implement a minimum dwell time constraint.

I.2 Motivation

This thesis started by asking how to improve methods for solving SOCPs, with a focus on a problem presented by Sonntag et al. in [1]. In it, a supermarket refrigeration system is controlled by simple on/off valves, with a changing number of compressors creating chilled fluid on one side and the display cases with the fluid cooling the air on the other side. The question they answered was how to optimize this system such that the set temperature was maintained in the display case while turning on/off the compressors and valves as little as possible. The developed method is to optimize the time the valves are open and the number of compressors that are on using a high-level Non-Linear Programming (NLP) solver in a non-linear Model Predictive Control (MPC) setting, resulting in an online iterative controller.

This system mixes continuous and discrete variables to make a switched system which cannot be solved in real time because of limits on computational power. They resolve this by splitting the original problem into separate problems, each with a different number of compressors running so each system is composed of only continuous variables and can be solved normally, and then the lowest cost solution among all solutions is used. Even doing it this way, the time taken each iteration is too great to be run in real time, so a low-level controller that does run in real time implements the valve opening based on the states of the system and the closing based on the given valve timings from the high-level controller; the high-level controller updates the timings and the number of compressors to turn on every time it finishes an iteration of the MPC. An issue with this method is that the solution has to sacrifice optimality because the solution to the problem cannot found in real time, so it cannot react to abrupt changes until the next iteration. In a thermal problem like the refrigeration system, the states change slowly enough with small enough outside interference that the sacrifice is acceptable, but faster changing problems can be unstable. The following paper attempts to address this by presenting a new approach to directly solve this type of problem with the benefit of implementing a minimum dwell time constraint which would be used in a system like the refrigeration system.

I.3 Literature Review

Since it can take a very long time to calculate the cost for every possible discrete control signal depending on the complexity of the system, several categories of simplifications have been researched to make solving SOCPs faster. The categories are setting the switching sequence and optimizing the timings, setting the timing and optimizing the switching, solving both at once, and finally by solving these problems in an MPC environment. Many methods for solving SOCPs are in the first category, since assuming the structure of the switching sequence means that only the timings of the switches are optimized, and since timings are continuous variables, this results in the problems being normal optimal control problems which can be solved with methods like the gradient method. In 1972 in a paper by Soliman and Ray [3], the gradient method is modified such that it can be used for these problems, and a paper by Aly and Chan [4] a year later uses a modified version of quasi-linearization of the dynamics and controls to solve. Edge and Powers [5] use function-space quasi-Newton algorithms with bounded controls to find the optimal solution in a bang-bang case where the optimal control is constrained and switches between its extrema. All of these papers handle a problem where the solution is on a singular arc wholly or in parts, which comes about whenever multiple solutions have the same cost, so solutions need to find a way to select one solution over another.

Later papers use the increased availability of computational power to utilize other methods that were out of reach before. Luus [6] uses iterative dynamic programming to find the solution when it is entirely on a singular boundary. Axelson et al. [7] use a problem where the state switches whenever it hits a boundary, and parametrize the timings of the switching state according to the initial state of the system. Sun [8] uses a line-up competition algorithm among the possible switches to find the optimal control, and Vossen [9] uses direct multiple shooting to find the optimal timings of the switching. Kamgarpour and Tomlin [10] use a similar parametrization of the switching times as before but prove guarantees on the possible cost and states of the system under the optimal control, and Jungers and Daafouz [11] find bounds on the cost when a dwell time is included and solved using periodic Lyapunov equations. Aronna, Bonnans, and Martinon [12] use indirect multiple shooting under a Gauss-Newton method to find optimal timings of the switching, and Heydari and Balakrishan [13] use adaptive dynamic programming to find the cost-to-go of the system dependent on the switching timings. Finally, Zhang et al. use neural networks to optimize the timings using dual heuristic programming.

The second category of problems where the the switching sequence is unknown but the timings are is very small, since there are few real world applications. One of the applications that generally is in this category is flight controls, since applying a global control to the vast mathematical model that can be an airplane is nigh-on impossible. Instead, researchers have broken the flight into separate modes, each with either different goals or with different sections of the state space, and an optimal controller determines when to switch between them. Schmidt, Garg, and Holowecky [15] use parameter optimization of the switching partitions between separate controllers to determine which controller is needed at a given time and position in the state space. Ferrari and Stengel [16] as well define controllers around specific points in the state space, using a neural network under adaptive dynamic programming to determine the optimal controller at each point.

The third category has come under great interest recently, as the increases in computing power have allowed researchers to solve both the optimal switching sequence and the timings of the switching at the same time. Generally this has been done hierarchically, where finding the sequence of the discrete controls is done by one controller and another finds the continuous controls given a discrete variable value, or it has been done by finding the value function of the switched system and solving that way. Dolcetta and Evans [17] find the value function of the system using dynamic programming with quasi-variational inequalities to solve. Hedlund and Rantzer [18] use a hybrid Bellman inequality to determine the lower bound on the cost of the system, and then use that to generate the optimal control. Xu and Antsaklis [19] use a two stage approach using dynamic programming to find an optimal switching sequence with one stage and then find the optimal control with that sequence fixed. Bengea and DeCarlo [20] instead use an embedding method to make the switched system into a normal system and solve that way. Borrelli et al. [21] find the value function and use that to determine a state feedback control law, and Rungger and Stursberg [22] use dynamic programming and dicretization of the state space and control to solve, and then bound the state errors between there discretization and the actual system. Englander, Conway, and Williams [23] again use a two stage approach, this time to find optimal interplanetary trajectories by optimizing the trajectory among pre-defined switching sequences in one stage and finding the optimal number of flybys in another. Kamgarpour et al. [24] also use a hierarchical controller similar to this.

Wardi, Egerstedt, and Hale [25] take a slightly different approach by discretizing the time and then optimizing the switching at each time point, using Armijo step sizing to determine which time instances to switch. Soler et al. [26] are also slightly different, using a relaxation technique similar to [20]. They multiply every mode of the system by binary-like controls that can vary between 0 and

1, and implement a constraint that says the sum of these controls always has to be 1. A cost function is then added that penalizes multiple controls being used at the same time, resulting in only one mode being fully active at a time. The new problem is then solved as a normal OCP. Heydari [27] returns to using the value function to solve, this time finding it using a function approximator. Finally, Foroozandeh, Shamsi, and Do Rosário De Pinho [28] use a combination of indirect shooting and the direct Euler method to solve without any *a priori* knowledge of the switching structure.

The fourth category of MPC is different from the other categories, because the control loop is iterative in real time. This generally means that at every time step, the MPC solves the SOCP over a finite time horizon, implements the first step of the controller, and then at the next time step runs again with the new states. The problem is that solving the MPC can often take much longer than a single time step, so not much research has been done in this area. Those that do generally solve this by using a hierarchical control, where a lower level controller in real time controls the system using parameters determined by the higher-level MPC, and the parameters update every time the MPC finishes an iteration. Sonntag et al. [1] in I.2 do exactly that. Deng et al. [29] also do this with a larger chiller plant on a campus, where one level uses dynamic programming to find the optimal response of the storage of the system to a load, and linearization and mixed-integer linear programming is used to find the optimal sequence of chillers that feed the storage. Shahsavari et al. [30] take a different approach, using a simplified version of the switching to find optimal switching times in an MPC.

The way to solve SOCPs explored below is to transform these problems into normal OCPs through the process of embedding as introduced in [20], adding a cost function, and then augmenting the state to formalize setting the switching frequency of the system. Chapter 2 will introduce the original SOCP and then discuss the embedding method, showing the disadvantages of this approach alone and giving an example. Chapter 3 introduces the new method that builds on the embedding approach by adding a cost to the control to guarantee that the solution found to the embedded problem can be directly utilized in the switched system, showing the same example with the new method. Chapter 4 details the augmentation of the system to make the switching frequency dependent on the problem setup, and shows a proof of this using a paper by Maurer [31]. Chapter 5 details the conclusions reached and future directions this research can go.

CHAPTER II

The Embedding Transformation

II.1 Switched Optimal Control Problem

The SOCP introduced by [20] is detailed first. Consider a switched autonomous dynamical system of the form:

$$\dot{x}(t) = f_{v(t)}(x(t), u(t)),\ x(t_0) = x_0\ \forall\forall t \in [t_0, t_f] \tag{II.1}$$

where $\forall\forall$ means "for almost every," $f_{v(\cdot)} \in \mathcal{C}^1(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$, and a cost functional of the form:

$$J(x(\cdot), u(\cdot), v(\cdot)) = \int_{t_0}^{t_f} L_{v(\tau)}(x(\tau), u(\tau))d\tau \tag{II.2}$$

where $L_{v(\cdot)} \in \mathcal{C}^1(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R})$ is the Lagrange cost. For the interval $[t_0, t_f]$, the control functions $v(\cdot)$ and $u(\cdot)$ must be chosen such that the following constraints are satisfied: $(t_0, x(t_0)) \in \mathcal{T}_0 \times \mathcal{B}_0 \subset \mathbb{R} \times \mathbb{R}^n$ and $(t_f, x(t_f)) \in \mathcal{T}_f \times \mathcal{B}_f \subset \mathbb{R} \times \mathbb{R}^n$, such that $t_0 < t_f$, where the endpoint constraint set $\mathcal{B} \triangleq \mathcal{T}_0 \times \mathcal{B}_0 \times \mathcal{T}_f \times \mathcal{B}_f$ is contained in a compact set. Then the switched optimal control problem is defined as:

$\min\limits_{u,v} J(x(\cdot), u(\cdot), v(\cdot))$ subject to:

(i) $x(\cdot)$ satisfies (II.1),

(ii) $(t_0, x(t_0), t_f, x(t_f)) \in \mathcal{B}$,

(iii) $v(t) \in \{0, 1\}, u(t) \in \Omega,\ \forall t \in [t_0, t_f]$, and

(iv) $\forall\ t_1, t_2 \in [t_0, t_f]$ with $v(t_1^-) \neq v(t_1^+)$ and $v(t_2^-) \neq v(t_2^+)$, $|t_1 - t_2| \geq T > 0$.

Constraint (iv) is a minimum dwell time constraint where the time elapsed between two changes in the value of the discrete decision variable, v, is required to be larger than a given constant $T > 0$. The minimum dwell time constraint puts an upper limit on the switching frequency of the system, and as such, solutions where the optimal control switches infinitely fast are not feasible. The minimum dwell time ensures that the optimal solution is implementable in the real system. Currently, formal methods do not exist to determine whether a solution is optimal or not when a minimum dwell time constraint is present, and the embedding method discussed in this chapter does not consider dwell-time constraints. While the problem above is formulated for an autonomous dynamical system with a time-independent Lagrange cost over a fixed finite time-horizon, extensions such as explicit time-dependence, variable time-horizon, and the presence of Mayer costs can be addressed using state-augmentation methods typically employed in optimal control. Furthermore, extensions to problems with multiple and/or integer-valued decision variables (as opposed to a single binary-valued discrete decision variable considered here) can be developed using techniques similar to [20].

II.2 The Embedding Process and the Embedded Optimal Control Problem

In order to make the SOCP solvable using gradient-based optimal control solvers, [20] embeds the switched system into a larger system where all decision variables are continuous. To embed the system, the discrete-valued control is replaced with a continuous-valued one that contains the original discrete values and then all the values in between. For the binary control, this means $v \in \{0, 1\}$ is transformed to $v_c \in [0, 1]$, where v_c can still be 0 or 1 at the extrema just like v, but can also take any other value in between 0 and 1. As a result, all decision variables in the system are continuous, and the SOCP is transformed into a normal OCP. The transformed embedded OCP (EOCP) considers the cost functional:

$$J(x(\cdot), u(\cdot), v_c(\cdot)) = \int_{t_0}^{t_f} [1 - v_c(\tau)]L_0(x(\tau), u(\tau)) + v_c(\tau)L_1(x(\tau), u(\tau))d\tau \tag{II.3}$$

and the dynamics:

$$\dot{x}(t) = [1 - v_c(t)]f_0(x(t), u(t)) + v_c(t)f_1(x(t), u(t)),\ x(t_0) = x_0,\ \forall t \in [t_0, t_f], \tag{II.4}$$

where $v_c \in [0, 1]$. The EOCP is formulated as:

$\min\limits_{u, v_c} J(x(\cdot), u(\cdot), v_c(\cdot))$ subject to:

(i) $x(\cdot)$ satisfies (II.4),

(ii) $(t_0, x(t_0), t_f, x(t_f)) \in \mathcal{B}$, and

(iii) $v_c(t) \in [0, 1],\ u(t) \in \Omega,\ \forall t \in [t_0, t_f]$.

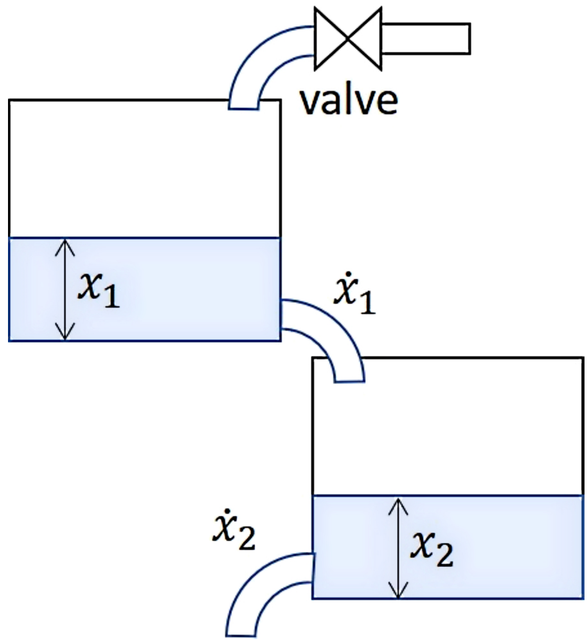

Figure II.1: The two tank system model used in the simulation.

II.3 Example

The following example illustrates the embedding process using a two-tank system. Two tanks are connected in series, with a fluid flowing through a valve into the first tank, which then outflows into a second tank, and then out of the system. The outflow from each tank is nonlinearly dependent on the square-root of the fluid level in the tank, and the goal is to maintain a specified level in the second tank by controlling the rate of flow into the first tank by opening or shutting the valve. The valve state is the discrete decision variable and can have either a high or a low flow rate. Fig. II.1 shows an overview of the system. The equations governing the fluid levels in the two tanks are:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{cases} \begin{bmatrix} 1 - \sqrt{x_1(t)} \\ \sqrt{x_1(t)} - \sqrt{x_2(t)} \end{bmatrix}, & v(t) = 0 \\ \begin{bmatrix} 2 - \sqrt{x_1(t)} \\ \sqrt{x_1(t)} - \sqrt{x_2(t)} \end{bmatrix}, & v(t) = 1 \end{cases} \quad \forall\forall t \in [t_0, t_f] \tag{II.5}$$

where x_1 is the fluid level in the first tank, x_2 is the fluid level in the second tank, and $v \in \{0, 1\}$ is the control. The cost functional for optimization is selected as:

$$J(x(\cdot), v(\cdot)) = \int_{t_0}^{t_f} \alpha(x_2(\tau) - 3)^2 d\tau \tag{II.6}$$

where $t_0 = 0$s, $t_f = 20$s, $\alpha = 2$, and $x(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ m. The goal is to achieve a fluid level of 3m in the second tank as defined in the cost functional, so the SOCP is formulated as:

$$\min_{u, v_c} J(x(\cdot), u(\cdot), v_c(\cdot)) \text{ subject to:}$$

(i) $x(\cdot)$ satisfies (II.5),

(ii) $(t_0, x(t_0), t_f, x(t_f)) \in \{0\} \times \begin{bmatrix} 2 \\ 2 \end{bmatrix} \times \{20\} \times \begin{bmatrix} (1,4) \\ (1,4) \end{bmatrix}$, and

(iii) $v(t) \in \{0, 1\}$, $\forall t \in [0, 20]$.

A constant fluid level of 3 m in the second tank would mean $\dot{x}_2(t) = 0$ m/s, so $\sqrt{x_1(t)} = \sqrt{x_2(t)} \implies x_1(t) = x_2(t) = 3$ m, so the level in the first tank would also be constant at 3 m. That means the flow rate into the first tank would need to be $\sqrt{3}$ m/s or about 1.73 m/s. This solution is between the high and the low flow rates of the valve, 2 m/s and 1 m/s, respectively, thus the ideal switched solution would be a sliding mode solution where the valve switches infinitely fast between the high and low values to achieve the desired rate. To show the sliding mode solution in the example used above, the numerical solution is found using the mode insertion method introduced in [25].

The mode insertion method utilizes a similar problem formulation as the SOCP, and as such, can be applied to the two-tank system. The time horizon is discretized, and an initial control sequence is introduced that is equal to 1 at every time instance. The change in the total cost in response to perturbation of the control signal at each time instance is calculated and the collection of time instances where the effect of control perturbations on the total cost is larger than a threshold is selected for modification. An iteration of the mode insertion technique then involves finding a combination of time instances that result in the lowest total cost. The control sequence from the said combination is then used in the next iteration. The process is repeated until the change in the total cost drops below another threshold. The first few iterations of this method work very quickly, dropping the total cost corresponding to the selected control sequences significantly. However, in following iterations the total cost decreases much slower, and approaches the optimal cost of the SOCP asymptotically.

The figure shown in Fig. II.2 represents the solution after 20 iterations. Continued iterations result in the switching happening more and more rapidly, eventually limited by the size of the time step

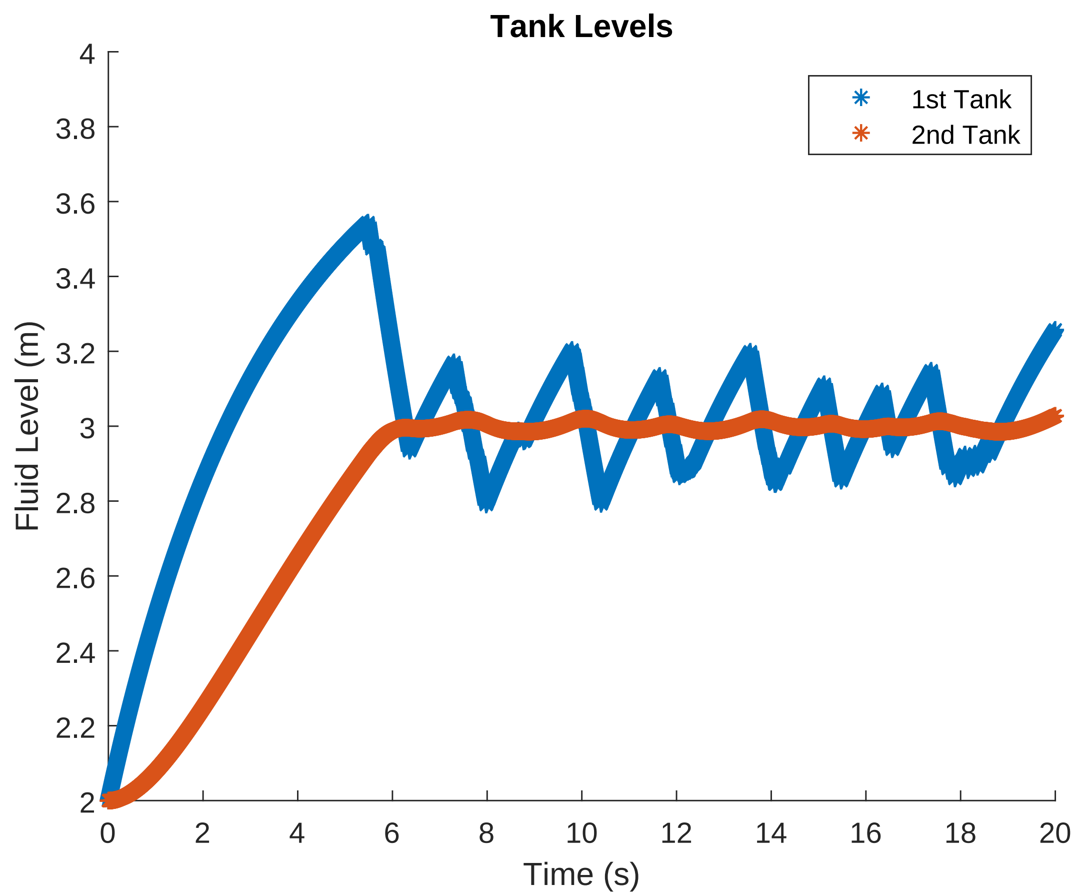

Figure II.2: The solution of the two tank system using the method of Ergestadt et al. [25]

of the discretization, which is consistent with the ideal optimal switched solution with a sliding mode-like behavior. The number of iterations directly affects the switching frequency, so dwell time constraints can only be implemented in this method by filtering the control signal in each iteration when choosing a new control sequence. The filtering method is, however, *ad hoc* with no guarantees on the final solution being optimal for the SOCP with a minimum dwell time constraint formulated in II.1.

When the two-tank example is solved using the embedding method, the cost function will stay the same as in the switched system,

$$J(x(\cdot), v_c(\cdot)) = \int_{t_0}^{t_f} \alpha(x_2(\tau) - 3)^2 d\tau, \tag{II.7}$$

but the dynamics of the system will be:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 1 + v_c(t) - \sqrt{x_1(t)} \\ \sqrt{x_1(t)} - \sqrt{x_2(t)} \end{bmatrix}, \ \forall t \in [0, 20] \tag{II.8}$$

where $v_c \in [0,1]$. Then the embedded optimal control problem is:

$\min_{v} J(x(\cdot), v(\cdot))$ subject to:

(i) $x(\cdot)$ satisfies (II.8),

(ii) $(t_0, x(t_0), t_f, x(t_f)) \in \{0\} \times \begin{bmatrix} 2 \\ 2 \end{bmatrix} \times \{20\} \times \begin{bmatrix} (1,4) \\ (1,4) \end{bmatrix}$, and

(iii) $v_c(t) \in [0,1]$, $\forall t \in [0,20]$.

In the following, a numerical solution to the EOCP computed using GPOPS is presented for comparison. GPOPS utilizes variable-order adaptive orthogonal collocation along with sparse nonlinear programming for finding numerically efficient solutions for OCPs. It also is very well documented, thus making it very transparent and easy to work with from a user's perspective. The optimal solution of the two-tank EOCP is presented in Fig. II.3 and II.4. This is the minimal cost achievable for the problem when the control is continuous because the system can now achieve the exact flow needed for the fluid level in the second tank to be a constant 3 m. However, since optimal flow rate is somewhere between the discrete high and low flow rates, the solution of the EOCP is not implementable on the original switched two tank system.

II.4 Limitations of Embedding

As stated earlier, the biggest limitation of the embedding transformation becomes apparent in those cases where the control needed to achieve the goal of the problem lies between two values of the discrete control variable, and as a result the embedded control can achieve the optimal value, but in the switched system the actual control cannot. If the optimal control lies on one of the extrema, which is often the case in time minimization problems, then the solutions to the embedded optimal control problem and the original switched optimal control problem can be shown to be identical [20, Proposition 5].

An unachievable control signal can be converted to one that is using the chattering lemma of Berkovitz in [33]. Using the chattering lemma, a switched solution can be found with a cost within ϵ of the embedded optimal cost, directly dependent on the switching frequency. The lower ϵ is, the faster the system will need to switch, where as ϵ approaches 0, the control switches infinitely fast (see Theorem 1 and its proof in Appendix A of [20]). However, there are several downsides to a

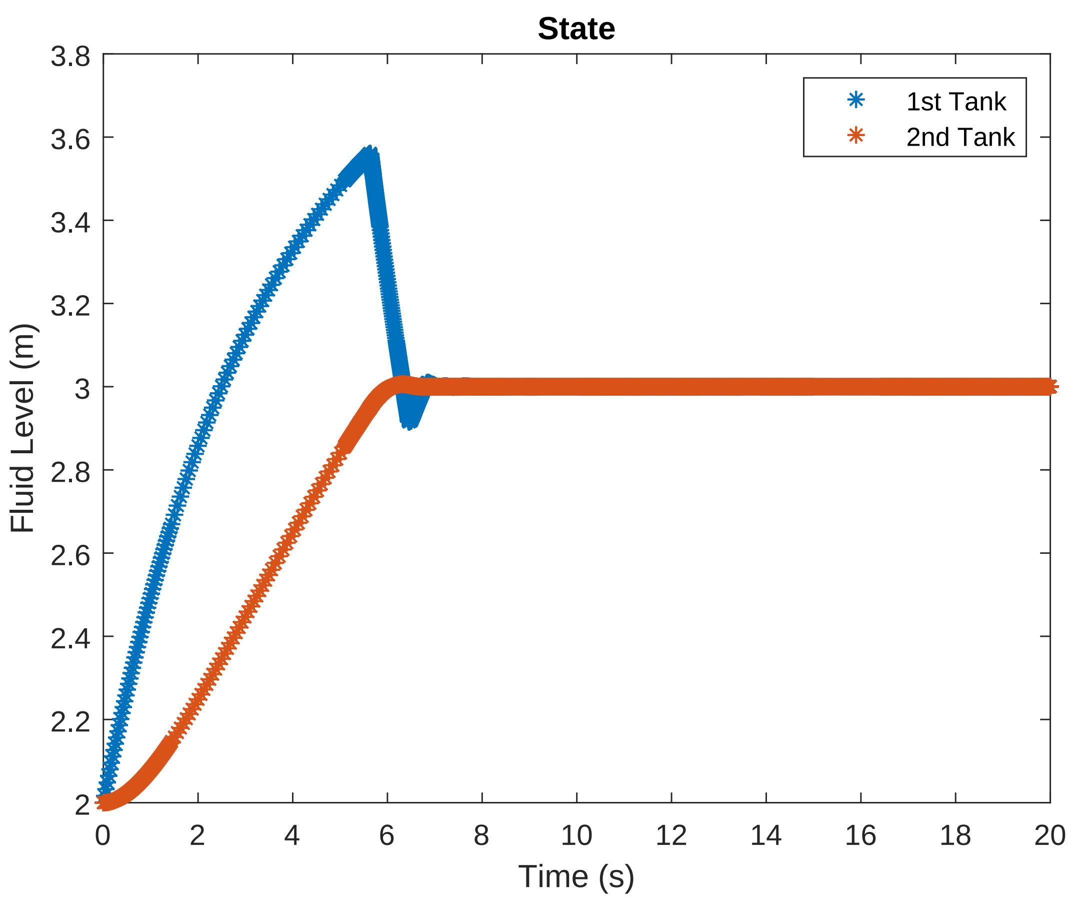

Figure II.3: The states of the two tank system using the embedding method of Bengea et al. [20]

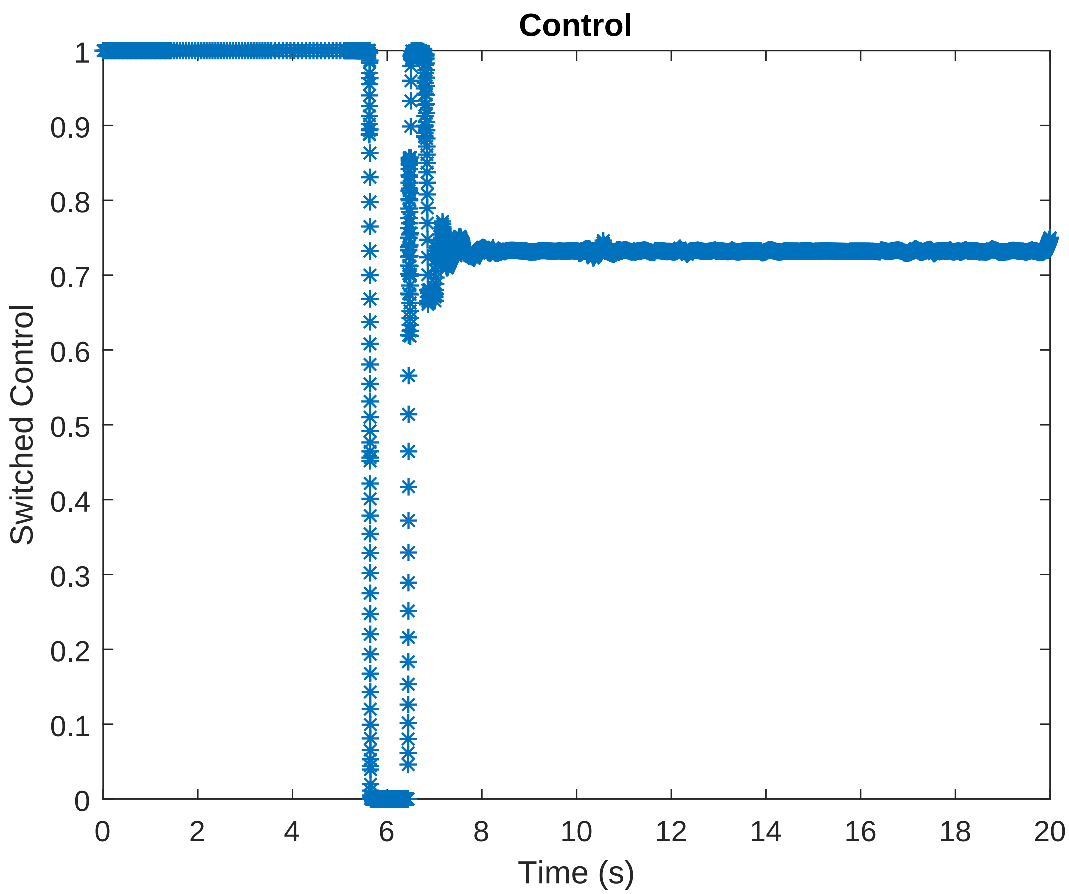

Figure II.4: The control of the two tank system using the embedding method of Bengea et al. [20]

chattering approach. The first is that using the chattering lemma has to be done after finding the solution, which adds computational time beyond solving the actual problem. More importantly, using the chattering lemma does not take into account any state constraints on the system, so the solution after using the chattering lemma may not be feasible for the original OCP without further modifications. A third downside is that minimum dwell time constraints are not included explicitly in the chattering lemma, so any application of such constraints would be *ad hoc*, through trial and error. The method developed in this thesis modifies the EOCP to ensure that a solution to the modified problem is directly implementable in the original switched system. The developed method can also satisfy minimum dwell time constraints by changing a weighting factor, making it applicable to many real world systems.

CHAPTER III

Modifications for Concavity of the Hamiltonian

III.1 Addition of a Cost Function on the Control

While embedding offers a convenient conduit for the application of gradient methods to optimize switched mode systems, singular solutions of the EOCP, owing to their sliding mode-like characteristics, are not realizable. For the solutions to be implementable, the optimal embedded control needs to be confined to the set of discrete values that the original switched control can take. That is, the goal of this chapter is to modify the EOCP so that the solution to the modified EOCP (MEOCP) is of a bang-bang type. To achieve the goal, an auxiliary cost is introduced on the new control variable such that the cost when the control variable is in the middle of its range is higher than when the cost is at its extrema, with the heuristic expectation that minimizing the cost will result in the control of the solution to the MEOCP being pushed to the boundary of its range. The auxiliary cost used in this chapter is an upside down parabola with zeros at the extrema, with a weight that scales the auxiliary cost relative to the original Lagrange cost. In particular, for the two-tank system, the auxiliary cost is $L_v(v_c(\cdot)) = \beta(v_c(\cdot) - v_c^2(\cdot))$, which outputs 0 when $v_c \in \{0, 1\}$, i.e. when the continuous control is in the range of the discrete control, and outputs a positive value when $v_c \in (0, 1)$.

Intuitively, a higher weight would result in a lower switching frequency in the solution, and vice versa, because the control switch will add to the overall cost scaled relative to the weighting, so when the weighting increases, the switching frequency would decrease to compensate, and vice versa. A minimum dwell time constraint can then be implemented by adjusting the weight of the auxiliary cost. In the following chapter, a detailed analysis of the MEOCP using Pontryagin's Minimum Principle (PMP) is presented along with numerical simulations for comparison.

III.2 Modified Optimal Control Problem

The MEOCP is almost identical to the EOCP, with the exception of the cost function, which now includes the auxiliary cost:

$$J(x(\cdot), u(\cdot), v_c(\cdot)) = \int_{t_0}^{t_f} [1 - v_c(\tau)]L_0(x(\tau), u(\tau)) + v_c(\tau)L_1(x(\tau), u(\tau)) + L_v(v_c(\tau))d\tau \qquad \text{(III.1)}$$

where $L_v \in \mathcal{C}^1((0,1), \mathbb{R}^+)$, $L_v(0) = L_v(1) = 0$. The dynamics are identical to those given previously in the embedded formulation, so the modified optimal control problem is:

$\min_{u,v_c} J(x(\cdot), u(\cdot), v_c(\cdot))$ subject to:

(i) $x(\cdot)$ satisfies (II.4),

(ii) $(t_0, x(t_0), t_f, x(t_f)) \in \mathcal{B}$, and

(iii) $v_c(t) \in [0,1]$, $u(t) \in \Omega$, $\forall t \in [t_0, t_f]$.

The numerical example below, with auxiliary cost $L_v(v_c(\cdot)) = \beta(v_c(\cdot) - v_c^2(\cdot))$ indicates that the solutions to the MEOCP are bang-bang, and show a correlation between the weighting factor, β, and the minimum dwell time.

III.3 Example

Consider the example used in II.3, with the dynamics:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 1 + v_c(t) - \sqrt{x_1(t)} \\ \sqrt{x_1(t)} - \sqrt{x_2(t)} \end{bmatrix}, \ \forall t \in [0, 20] \qquad \text{(III.2)}$$

and the modified cost function:

$$J(x(\cdot), v_c(\cdot)) = \int_{t_0}^{t_f} \alpha(x_2(\tau) - 3)^2 + \beta(v_c(\tau) - v_c^2(\tau))d\tau. \qquad \text{(III.3)}$$

The statement of the MEOCP is identical to the statement of the EOCP, and is excluded for brevity. As anticipated, the numerical optimal solution found using GPOPS is of a bang-bang type and the dwell time is directly related to the value of the weight β in the auxiliary cost L_v. The correlation between the weight and the switching frequency of the control can be seen in Fig. III.1 and III.2 with $\beta = 0.001$ and Fig. III.3 and III.4 with $\beta = 0.5$. When the weight is greater, the switching frequency is lower, the solution stays at the extremes longer, and the level in the second tank varies

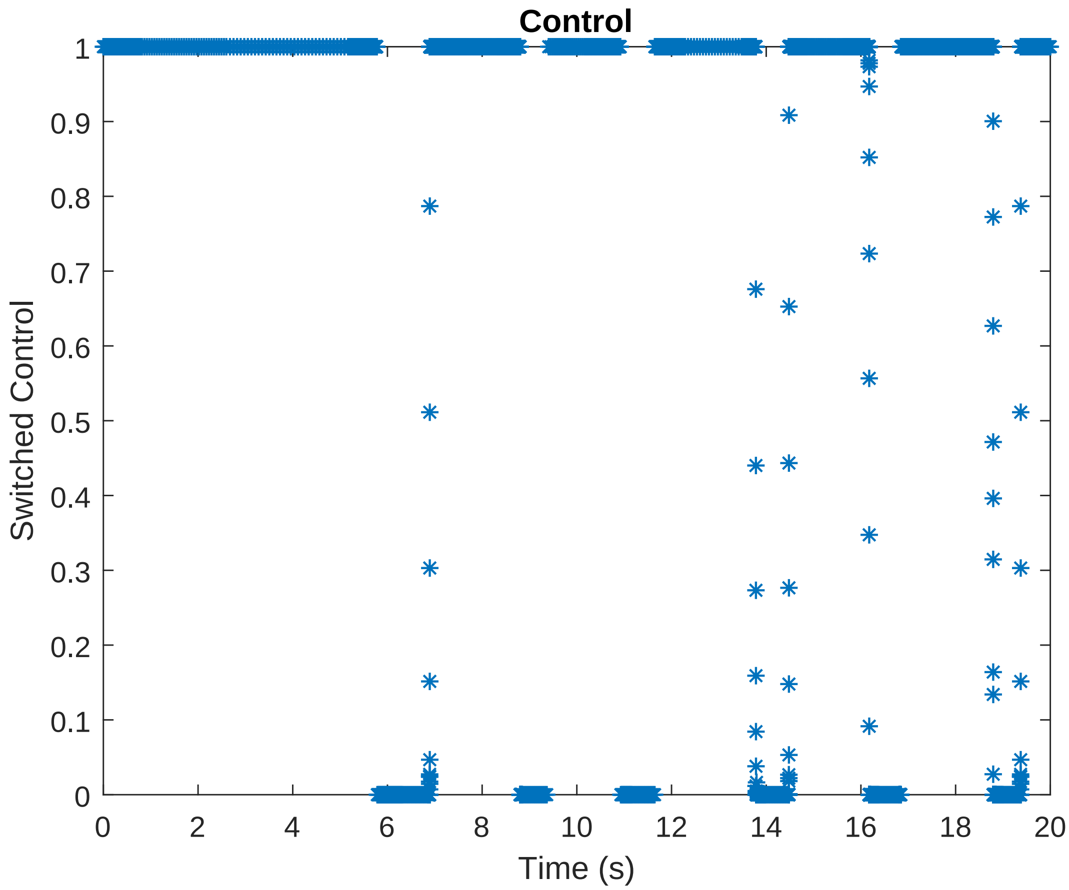

Figure III.1: The optimal control signal of the optimal solution to the two tank system using the modified embedding method and a weight of 0.01.

farther from the goal, and vice versa. In the following section, the MEOCP is analyzed using PMP to explore theoretical consistency of the experimental observations.

III.4 Pontryagin's Minimum Principle

PMP is a foundational result in optimal control that has resulted in a multitude of tools that transform an infinite-dimensional optimization problem into a much more computationally efficient two-point boundary value problem formulated on a Hamiltonian system of differential equations that depend on the state variables, the control, the cost function, and a new set of variables called the costate. PMP states that the control signal that results in an optimal cost will also be constrained by the Hamiltonian system of differential equations while satisfying pre-specified constraints on the states and the control, along with a new set of constraints on the costate, commonly known

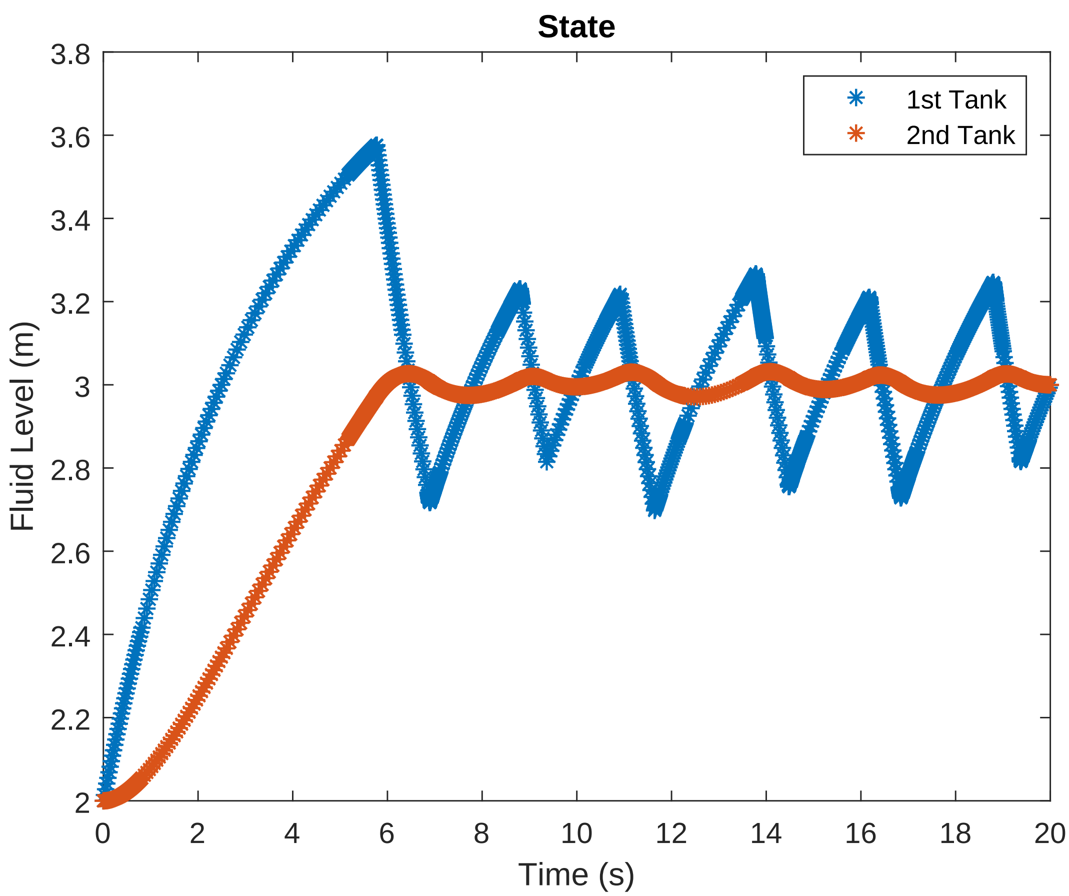

Figure III.2: The optimal solution of the two tank system using the modified embedding method and a weight of 0.01.

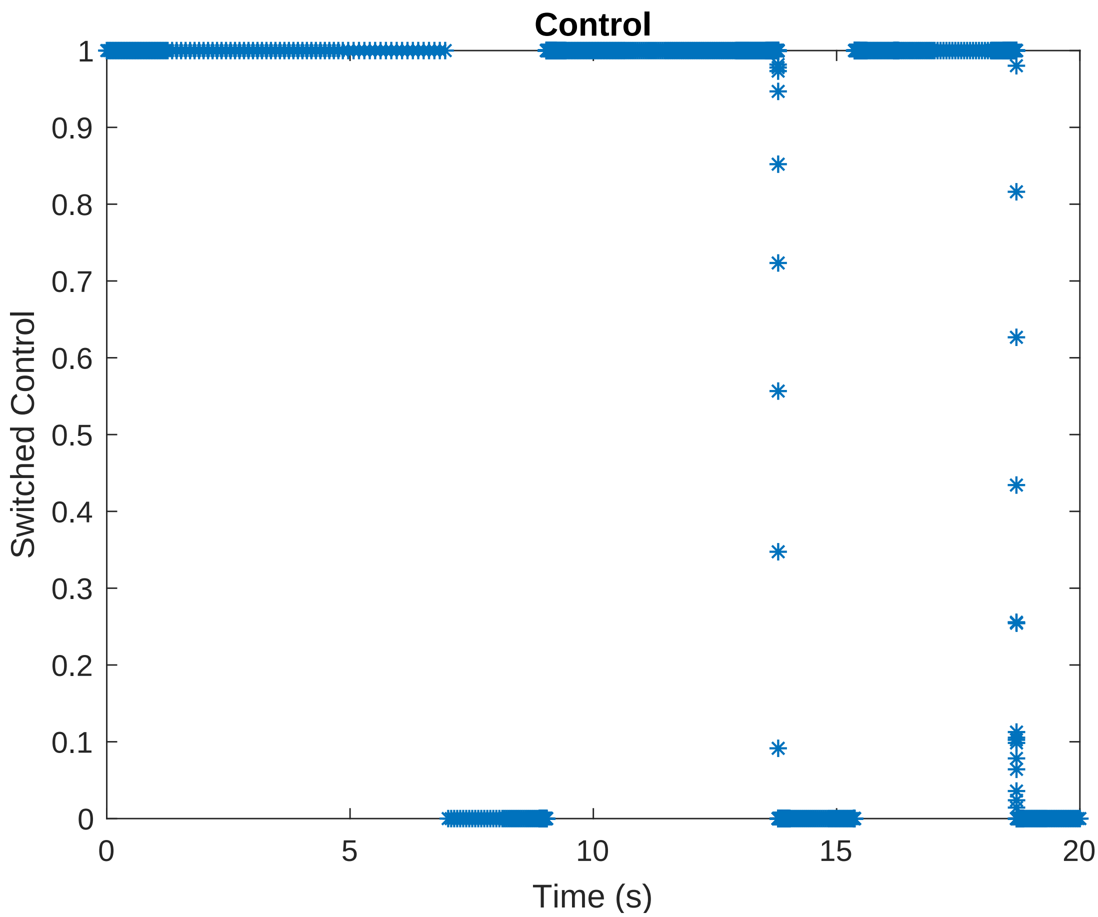

Figure III.3: The control signal of the optimal solution of the two tank system using the modified embedding method and a weight of 0.5.

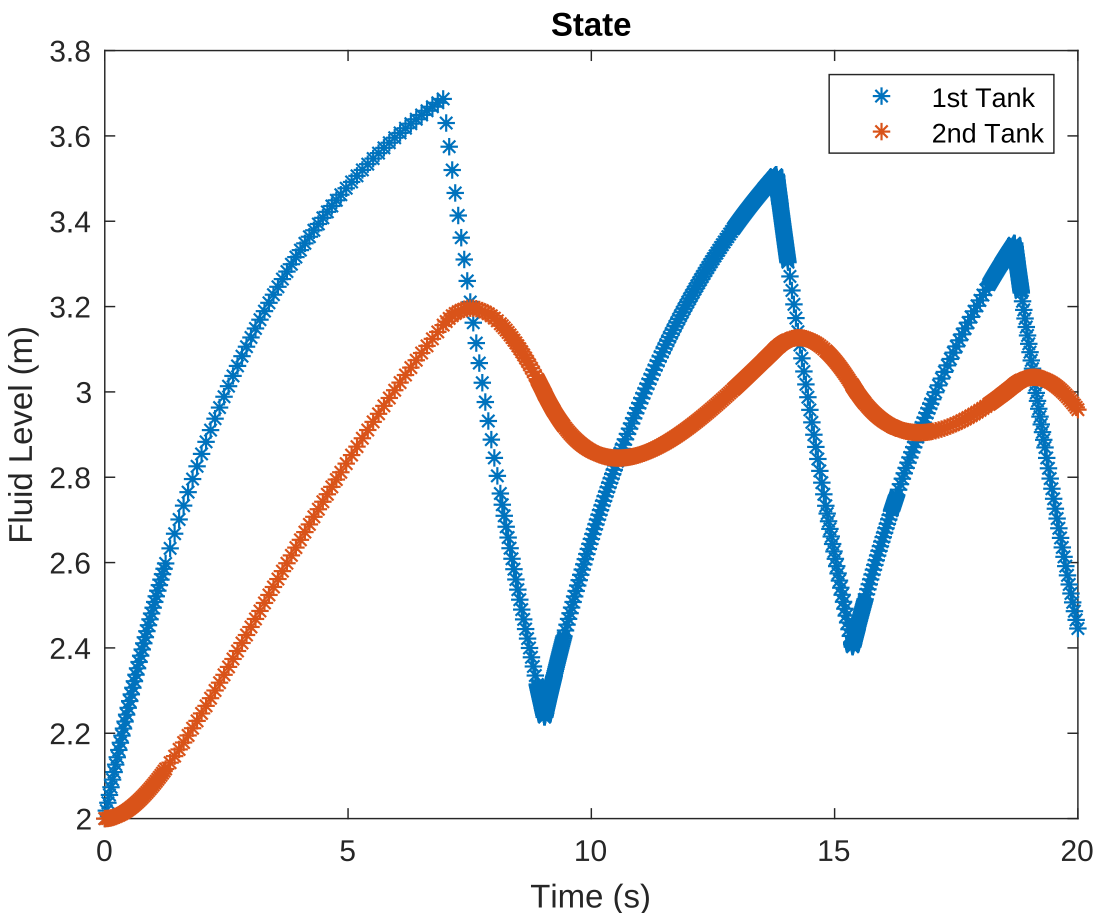

Figure III.4: The optimal solution of the two tank system using the modified embedding method and a weight of 0.5.

as the transversality conditions. Consider a dynamical system with states $x \in \mathbb{R}^n$ and a control $u \in \mathcal{U} \subset \mathbb{R}^m$ with the dynamics:

$$\dot{x}(t) = f(x(t), u(t)),\ x(t_0) \in \mathbb{R}^n \forall t \in [t_0, t_f] \tag{III.4}$$

and the cost function:

$$J(x(\cdot), u(\cdot)) = \int_{t_0}^{t_f} L(x(\tau), u(\tau)) d\tau \tag{III.5}$$

where $f \in \mathcal{C}^1(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$ and $L \in \mathcal{C}^1(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R})$. For the costates $p(\cdot) \in \mathbb{R}^n$, the Hamiltonian is defined as:

$$H(x(\cdot), p(\cdot), u(\cdot)) = p^{\mathsf{T}}(\cdot) f(x(\cdot), u(\cdot)) + L(x(\cdot), u(\cdot)). \tag{III.6}$$

Then along the optimal state trajectory $x^*(\cdot)$, the optimal control trajectory $u^*(\cdot)$, and the optimal costate trajectory $p^*(\cdot)$, the Hamiltonian satisfies the following necessary conditions:

(i) Hamiltonian minimization: $H(x^*(t), p^*(t), u^*(t)) \leq H(x^*(t), p^*(t), u) \quad \forall\, u \in \mathcal{U}$ and $t \in [t_0, t_f]$,

(ii) Costate dynamics: $\dot{p}^{\mathsf{T}}(\cdot) = -H_x(x^*(\cdot), p(\cdot), u^*(\cdot))$,

(iii) $H(t_f) = 0$, and

(iv) the transversality conditions: $p(t_f) = \mathbf{0}^n$,

where the subscript x represents partial differentiation with respect to the states and $\mathbf{0}^n$ represents an n-vector of zeros. If the state of the system at the final time is fixed, the transversality condition does not apply and the constraints on the final condition replace it to yield a well-defined two-point boundary problem.

III.5 Limitations of Modification

Theorem III.1. *If an EOCP as defined above has a Hamiltonian formulation that is concave with respect to the control signal, then the optimal control satisfies* $v_c^*(t) \in \{0, 1\}\ \forall t \in [t_0, t_f]$)

Proof. When PMP is applied to the embedded problem, the Hamiltonian is:

$$\begin{aligned} H(x(\cdot), p(\cdot), u(\cdot), v_c(\cdot)) = p^{\mathsf{T}}(\cdot)[(1 - v_c(\cdot)) f_0(x(\cdot), u(\cdot)) + v_c(\cdot) f_1(x(\cdot), u(\cdot))] + \\ (1 - v_c(\cdot)) L_0(x(\cdot), u(\cdot)) + v_c(\cdot) L_1(x(\cdot), u(\cdot)) + L_v(v_c(\cdot)) \end{aligned} \tag{III.7}$$

and the dynamics of the costates are:

$$\dot{p}^{\intercal}(\cdot) = -p^{\intercal}(\cdot)\frac{\partial}{\partial x}[(1-v_c(\cdot))f_0(x(\cdot),u(\cdot)) + v_c(\cdot)f_1(x(\cdot),u(\cdot))] + \frac{\partial}{\partial x}[(1-v_c(\cdot))L_0(x(\cdot),u(\cdot)) + v_c(\cdot)L_1(x(\cdot),u(\cdot))]. \quad \text{(III.8)}$$

In this case, the Hamiltonian minimization condition implies that:

$$H(x^*(t),p^*(t),u^*(t),v_c^*(t)) \leq H(x^*(t),p^*(t),u,v_c) \ \forall\ u \in \mathcal{U},\ v_c \in [0,1]. \quad \text{(III.9)}$$

The function $v_c \mapsto H(x,p,u,v_c)$ is the sum of an affine function $v_c \mapsto p^{\intercal} f_0(x,u) + L_0(x,u) + v_c(-p^{\intercal} f_0(x,u) - L_0(x,u) + f_1(x,u) + L_1(x,u))$ and a concave function $v_c \mapsto L_v(v_c)$, and as a result is concave. Specifically, for $L_v(v_c) \coloneqq \beta(v_c - v_c^2)$,

$$\frac{\partial^2}{\partial v_c^2}H(x,p,u,v_c) = \frac{\partial^2}{\partial v_c^2}L_v(v_c) = -2\beta. \quad \text{(III.10)}$$

As a result, minimization of the Hamiltonian with respect to v_c results in $v_c^*(t) \in \{0,1\}\ \forall\, t \in [t_0,t_f]$. That is, the optimal embedded control is bang-bang. □

Note that since the auxiliary cost evaluates to zero when $v_c(\cdot) \in \{0,1\}$, Theorem III.1 implies that the auxiliary cost is identically zero along any optimal solution of the MEOCP. As a result, the total cost along the optimal solution is independent of β. Furthermore, as evidenced by equation III.8, the costate dynamics, and hence, the optimal control trajectory, are independent of L_v beyond the auxiliary cost-induced forcing of v_c to the boundary of its range. In particular, the PMP fails to explain the connection between β and the minimum dwell time. In fact, a heuristic treatment of the two-tank system indicates that a sliding mode solution where infinite switching frequency is utilized to maintain the desired fluid level would be optimal for any $\beta > 0$.

In the numerical solution, the solver uses orthogonal collocation which selects a set of points for evaluation, and in its single-phase form, is designed for continuous decision variables. In this case, it interpolates between discrete values of the bang-bang control. Thus, whenever the control switches, its numerical representation is continuously connected between the pre- and post-switching values, making the control signal traverse the interior of its range. As a result, the auxiliary cost has a nonzero contribution in the total cost of the trajectory. The contribution due to $L_v(\cdot)$ increases with the number of switches and also with increasing β. We postulate that an increase in β is compensated by a decrease in the number of switches, resulting in the behavior observed in the numerical solution detailed in figures III.2 and III.4.

While multi-phase implementations of pseudospectral optimization are capable of handling discontinuous decision variables, the timing of the switches is unknown *a priori*, and as a result the solver is unable to split the time horizon exactly at the time of each switch and interpolation still is required. As a result, the solution is still dependent on the weight β. To test this postulate, a very simple problem of a particle accelerating and then stopping with a known discontinuity at the halfway point was tried with the proposed method. The dynamics of this system with states $x \in \mathbb{R}^2$ and control $u \in [-1, 1]$ are:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ u(t) \end{bmatrix}, \ \forall t \in [t_0, t_f] \tag{III.11}$$

where the cost function is:

$$J(x(\cdot), u(\cdot)) = \int_{t_0}^{t_f} (x_1(\tau)^2 + \gamma(1 - u(\tau)^2))d\tau \tag{III.12}$$

The optimal control problem is:

$\min_{u} J(x(\cdot), u(\cdot))$ subject to:

(i) $x(\cdot)$ satisfies (III.11),

(ii) $(t_0, x(t_0), t_f, x(t_f)) \in \{0\} \times \begin{bmatrix} 1 \\ 0 \end{bmatrix} \times \{2\} \times \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and

(iii) $u(t) \in [-1, 1]$, $\forall t \in [t_0, t_f]$.

When the solver is setup to split the time horizon into 2 phases directly at the discontinuity, the cost dependence on the weight γ is no longer seen. The optimal control and states for this example are shown in Fig. III.5 and III.6.

When the solver is forced to use a single phase over the entire time, the optimal solution still looks the same as before. As evidenced by Table III.1, the overall cost becomes dependent on the weight γ of the auxiliary cost function. The numerical experiment described above supports the postulate that the dependence seen between dwell time and the auxiliary cost weight is an artifact of the numerical solver and is not a feature of the problem itself. As a result, there is no solver-independent way to compute the auxiliary cost weight that will achieve a given dwell-time. To formalize the dependence of the dwell time on the auxiliary cost, the next chapter examines a state-augmented form of the MEOCP.

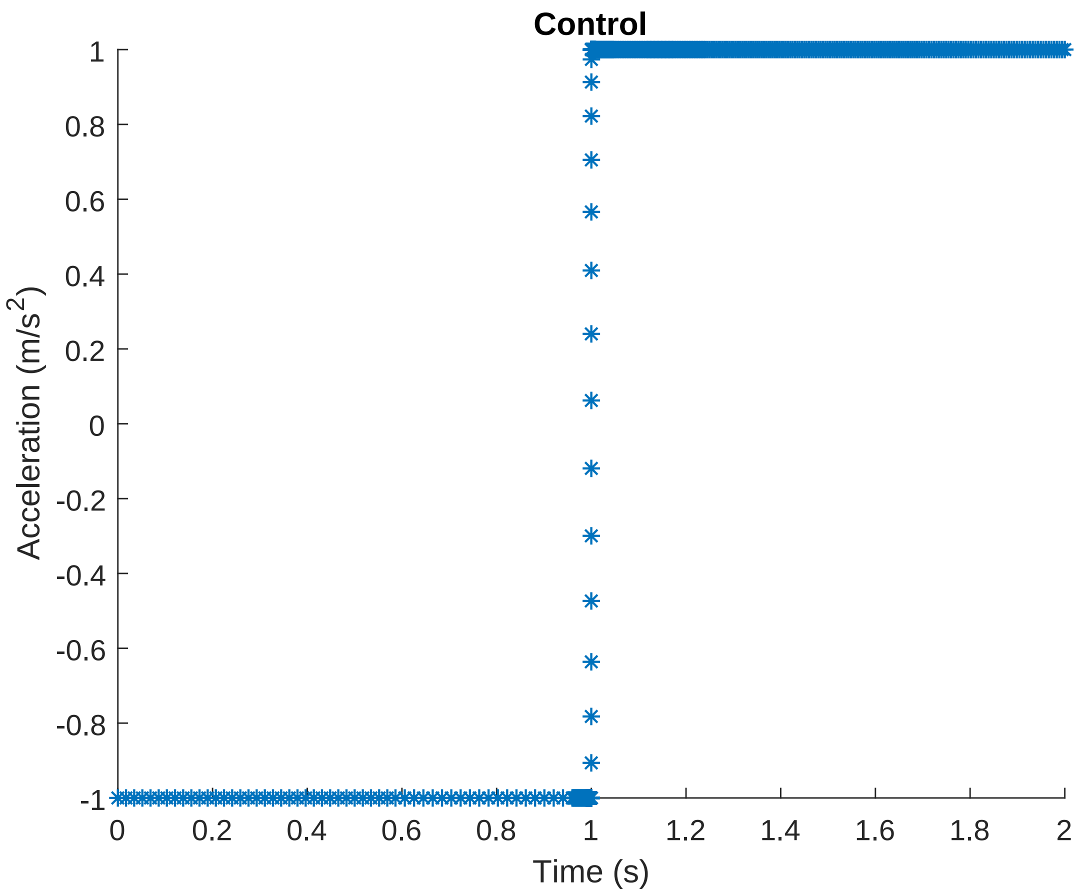

Figure III.5: The control signal of the optimal solution to the particle acceleration problem and is identical across all auxiliary cost weights.

Table III.1: Effect of weighting and number of phases on total cost.

β	1 Phase	2 Phases
0.001	0.76669	0.76666
0.01	0.76690	0.76666
0.1	0.76903	0.76666
1	0.79028	0.76666
10	1.00279	0.76666

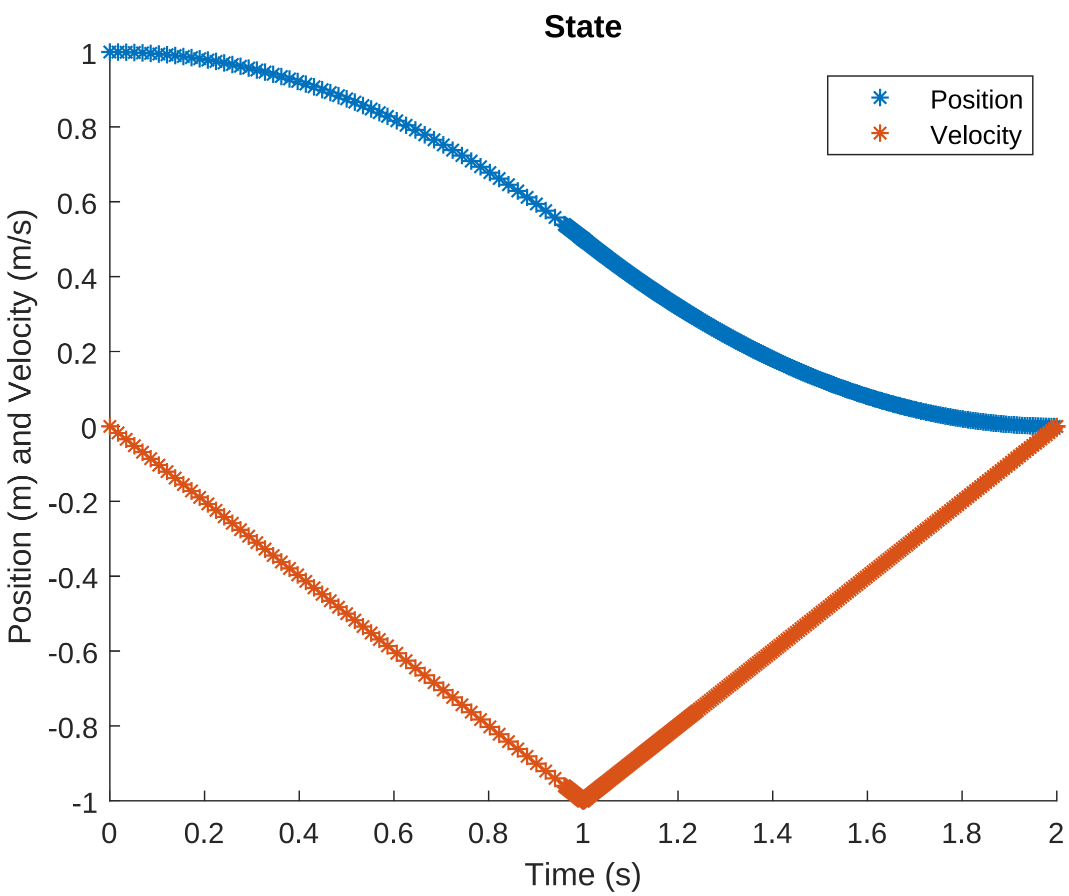

Figure III.6: The optimal solution of the particle acceleration problem and is identical across all auxiliary cost weights.

CHAPTER IV

Analysis of Numerical Artifacts in the Embedded Problem

IV.1 Addition of a State

Pontryagin's Minimum Principle, when applied to the MEOCP, is unable to explain the relation between the dwell time of the numerical solution and the weighting on the auxiliary cost due to the discontinuities in the optimal control signal. This chapter aims to formalize the connection between the auxiliary cost and the dwell time using an augmented modified optimal control problem (AMEOCP). In the AMEOCP, the dynamics are augmented by treating the embedded control signal of the MEOCP as a state, i.e. $x_{n+1}(\cdot) = v_c(\cdot)$.

The augmented stated is assumed to be the integral of $\gamma w(\cdot)$, where $\gamma > 0$ is a constant and $w(\cdot)$ is the new control signal of the AMEOCP. For a MEOCP with a single discrete decision variable, the augmented dynamics are given by:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x}_{n+1}(t) \end{bmatrix} = \begin{bmatrix} [1 - x_{n+1}(t)]f_0(x(t), u(t)) + x_{n+1}(t)f_1(x(t), u(t)) \\ \gamma w(t) \end{bmatrix} X(t_0) = X_0 \; \forall t \in [t_0, t_f]. \tag{IV.1}$$

The bounds on x_{n+1} are the same as the bounds on the embedded control signal v_c, i.e. $x_{n+1}(t) \in [0, 1]$, $\forall\, t \in [t_0, t_f]$. The new control signal $w(\cdot)$ satisfies $w(t) \in [-1, 1]$, $\forall\, t \in [t_0, t_f]$.

The AMEOCP is motivated by the observation that by passing it through an integrator, the theoretical bang-bang behavior of $v_c(\cdot)$ can be modified to better suit numerical computations, where the transitions are smooth. The new control signal $w(\cdot)$ then selects the slope of the smooth transitions of $v_c(\cdot)$ (i.e. $x_{n+1}(\cdot)$) as a fraction of the constant γ. The cost functional of the AMEOCP is given by:

$$J(X(\cdot), u(\cdot), w(\cdot)) = \int_{t_0}^{t_f} [1 - x_{n+1}(\tau)]L_0(x(\tau), u(\tau)) + x_{n+1}(\tau)L_1(x(\tau), u(\tau)) + L_v(x_{n+1}(\tau))d\tau \tag{IV.2}$$

The AMEOCP is then formulated as

$\min_{u,w} J(X(\cdot), u(\cdot), w(\cdot))$ subject to:

(i) $X(\cdot)$ satisfies (IV.1),

(ii) $(t_0, x(t_0), t_f, x(t_f)) \in \mathcal{B}$,

(iii) $(x_{n+1}(t_0), x_{n+1}(t_f)) \in [0,1] \times [0,1]$, and

(iv) $w(t) \in [-1,1]$, $u(t) \in \Omega$, $\forall t \in [t_0, t_f]$.

Note that if the systems f_0 and f_1 were affine with respect to the control signal $u(\cdot)$, then the augmented system is affine with respect to the control $\begin{bmatrix} u(\cdot) \\ w(\cdot) \end{bmatrix}$. Since there are bounds on x_{n+1}, the AMEOCP is a *path constrained* optimal control problem, and as such, is difficult to analyze in general. Fortunately, the AMEOCP dynamics are control-affine, which allows for the use of results specific to control-affine systems. In the following, the AMEOCP is analyzed using Maurer's formulation of PMP for constrained optimal control problems with control appearing linearly [31].

IV.2 Proof of Solution to Problem using Maurer

While many of the assumptions are the same between the AMEOCP and the formulation in [31], there are a few major differences that need to be addressed. The first is that [31] addresses problems with controls that appear linearly in the system. While the augmented system is linear with respect to $w(\cdot)$, linearity with respect to $u(\cdot)$ only if the original switched system was linear with respect to $u(\cdot)$. Analysis of the AMEOCP for switched systems where $u(\cdot)$ appears nonlinearly require alternative formulations of the PMP and is outside the scope of this thesis. Alternatively, the continuous controls $u(\cdot)$ can be found *a priori* for each operating mode and the method developed herein can be used to determine the sequence and the timing of mode switches.

The second difference is that the AMEOCP is formulated using a Lagrange cost, which is an integral of a Lagrange cost function over the time horizon of the optimal control problem. In [31], the cost is instead a Mayer cost, which is a function of the final state of the system. In order to apply the PMP from [31] to the AMEOCP the dynamics are augmented again with a state that represents the cost, which starts at 0 with a time-derivative equal to the Lagrange cost function. The Mayer cost is then given by the value of the new augmented state at the final time, which is equivalent to the Lagrange cost.

In the problem introduced by Maurer, the states are $x(\cdot) \in \mathbb{R}^n$, the control $u(\cdot)$ is linear, scalar, and piecewise continuous, and the dynamics of the system are:

$$\dot{x}(t) = f(x(t), u(t)) \; \forall t \in [t_0, t_f], \;\; x_0(t_0) \in \mathbb{R}^n. \tag{IV.3}$$

where $f(\cdot) \in \mathcal{C}^1(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$. The control and states have to fulfill the inequality constraints:

$$|u(t)| < K(t) \mid K(t) > 0, \;\; S(x) \leq \delta \mid \delta \in \mathbb{R}, \;\; \forall t \in [t_0, t_f] \tag{IV.4}$$

where $K(\cdot) \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$, $S(\cdot) \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$, and $\delta \in \mathbb{R}$. The cost function for this problem is in Mayer form, where:

$$J(x(\cdot), u(\cdot)) = G(x(t_f)) \tag{IV.5}$$

where $G(\cdot) \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$.

For an AMEOCP formulated using a Mayer cost for a switched system without continuous control variables (e.g. the two-tank system), the dynamics are:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x}_{n+1}(t) \\ \dot{x}_{n+2}(t) \end{bmatrix} = \begin{bmatrix} [1 - x_{n+1}(t)]f_0(x(t)) + x_{n+1}(t)f_1(x(t)) \\ \gamma w(t) \\ [1 - x_{n+1}(t)]L_0(x(t)) + x_{n+1}(t)L_1(x(t)) + L_v(x_{n+1}(t)) \end{bmatrix} \tag{IV.6}$$

where $\mathcal{X}(t) \in \mathbb{R}^n \times [0,1] \times \mathbb{R}^+$, and $\mathcal{X}(t_0) = \mathcal{X}_0 \; \forall t \in [t_0, t_f]$. The control constraint is $|w(\cdot)| \leq K(\cdot) \equiv 1$. Since the only state constraint is on the first augmented control and this needs to be between the original discrete control values of 0 and 1, the function $S(\cdot)$ is selected as $S(x_{n+1}(\cdot)) = x_{n+1}^2(\cdot) - x_{n+1}(\cdot) \leq \alpha$, where $\alpha = 0$. This function is used because it is the exact inverse of the auxiliary cost function, and as a result, it simplifies the calculations. Using this function, $S(x_{n+1}(t)) = 0$ when $x_{n+1}(t) \in \{0, 1\}$, i.e. the augmented state is on the boundary, $S(x_{n+1}(t)) < 0$ whenever $x_{n+1}(t) \in (0, 1)$, i.e. the augmented state is in the interior of its range. Maurer then splits the dynamics into two sets of equations, one that does not contain the linear control, and one that does, with the control factored out. That is,

$$\dot{\mathcal{X}}(t) = g(X(t)) + [\mathbf{0}^n \; \gamma \; 0]^\intercal w(t) \tag{IV.7}$$

where $g(X(\cdot)) \in \mathbb{R}^{n+2}$ and

$$g(X(t)) = \begin{bmatrix} [1 - x_{n+1}(t)]f_0(x(t)) + x_{n+1}(t)f_1(x(t)) \\ 0 \\ [1 - x_{n+1}(t)]L_0(x(t)) + x_{n+1}(t)L_1(x(t)) + L_v(x_{n+1}(t)) \end{bmatrix}, \forall \; t \in [t_0, t_f].$$

The cost function is $J(\mathcal{X}(t_f)) = x_{n+2}(t_f)$, which is the final state of the augmented cost.

Define the Hamiltonian as:

$$H(\mathcal{X}(\cdot), \mathcal{P}(\cdot), w(\cdot), \eta(\cdot)) = \mathcal{P}^T(\cdot)g(X(\cdot)) + \gamma p_{n+1}(\cdot)w(\cdot) + \eta(\cdot)S(x_{n+1}(\cdot)) \tag{IV.8}$$

where $\mathcal{P} \in \mathbb{R}^{n+2}$ are the costates and $\eta \in \mathbb{R}$ is the switching function. The necessary conditions of Pontryagin's Minimum Principle are:

(i) There exists a function $\eta(\cdot)$ such that the costates in this problem satisfy:

$$\dot{p}^T(t) = -\mathcal{P}^T(t)g_x(X(t))$$

$$\dot{p}_{n+1}(t) = -\mathcal{P}^T(t)g_{x_{n+1}}(X(t)) - \eta(t)S_{x_{n+1}}(x_{n+1}(t))$$

$$\dot{p}_{n+2}(t) = 0 \;\forall t \in [t_0, t_f]$$

$$p(t_f) = \mathbf{0}^n$$

$$p_{n+1}(t_f) = \eta_0$$

$$p_{n+2}(t_f) = 0$$

where the subscript on a function represents the partial derivative according to that variable.

(ii) The function $\eta(\cdot)$ satisfies $\eta(t)S(x_{n+1}(t)) \equiv 0 \;\forall t \in [t_0, t_f]$

(iii) $H(\mathcal{X}^*(t), \mathcal{P}^*(t), w^*(t), \eta^*(t)) \leq H(\mathcal{X}^*(t), \mathcal{P}^*(t), w(t), \eta^*(t)) \;\forall\, w(t) \in [-1, 1]$ and $t \in [t_0, t_f]$ where $\mathcal{X}^*$, $\mathcal{P}^*$, w^*, and η^* denote the optimal state, costate, and control trajectories.

The coefficient of $w(\cdot)$ in IV.8 is called the switching function, $\phi(\cdot) = \gamma p_{n+1}(\cdot) \in \mathbb{R}$. The switching function determines the optimal control of the system. The optimal solution to the problem in IV.6 is compromised of three types of arcs dependent on whether or not the system is on the boundary of the augmented state constraint [31].

When the system is on a boundary arc, that is when $x_{n+1}(\cdot) \in \{0, 1\}$, the optimal control is $w(\cdot) = 0$, because the control signal is constant while the system is at either boundary of its range. Let the control be analytic on a subinterval $I \subset [t_0, t_f]$ and let $x(t)$ be an analytic solution on I. Define the functions $\varphi_i : I \to \mathbb{R}^n$ recursively such that:

$$\varphi_0(t) = f_u(t), \;\; \varphi_{i+1}(t) = \dot{\varphi}_i(t) - f_x(t)\varphi_i(t), \;\; i \geq 0 \tag{IV.9}$$

Then the following relations hold on I:

$$\phi(t) = \mathcal{P}^T(t)\varphi(t)_0 = 0,$$
$$\dot{\phi}(t) = \mathcal{P}^T(t)\varphi(t)_1 - \eta(t)b(t) = 0 \quad \forall\ t \in I$$

where $b(\cdot) = \gamma S_{x_{n+1}}(x_{n+1}(\cdot))$. In this system, the state constraint uses the augmented state $x_{n+1}(\cdot)$ that is explicitly controlled by $w(\cdot)$ in its dynamics, so according to Maurer only the first derivative of the function ϕ is defined.

When $x_{n+1}(\cdot)$ is in the interior of its range, the optimal trajectories can be in one of the other two different arcs. The first is an interior singular arc, which means $\phi(\cdot) \equiv 0$ and thus $p_{n+1}(\cdot) \equiv 0$, so $\dot{p}_{n+1}(\cdot) \equiv 0$. Since this arc is on the interior, $S(x_{n+1})$ is nonzero, so $\eta(\cdot)$ must be 0 to fulfill necessary condition ii, which leaves $\mathcal{P}^T(t)g_{x_{n+1}}(X(t)) \equiv 0$. This means that either the function $g(\cdot)$ is unchanging with respect to x_{n+1} or that the costates are 0. Since x_{n+1} is directly in g, the first cannot hold, but if the costates are zero, then the states of the system cannot be changing and $\dot{\mathcal{X}} \equiv 0$. Then all derivatives of states and costates would be zero, which is assumed to not be the case and thus the system can never be in an interior singular arc.

The other possible arc is an interior nonsingular arc, that is the optimal trajectory is not at the boundaries of its range and also analytic. The switching function is equal to:

$$\phi(\cdot)^{(i)} = xP^T(\cdot)\varphi(\cdot)_{(i)},\ i = 0, 1$$

where the equation $\varphi(\cdot) \in \mathbb{R}^{n+2}$ is defined recursively as:

$$\varphi_0(\cdot) = [\mathbf{0}^n\ \gamma\ 0]^T,$$
$$\varphi_1(\cdot) = \dot{\varphi}_0(\cdot) - g_{\mathcal{X}}(\cdot)\varphi_0(\cdot)$$

and the superscript is the time derivative and the subscript is the partial derivatives according to $\mathcal{X}$ if not specified otherwise. In an AMEOCP, $\phi(\cdot) = \gamma p_{n+1}(\cdot)$ and $\dot{\phi}(\cdot) = -\gamma\mathcal{P}^T g_{x_{n+1}}(X(\cdot))$. Since H is linear with respect to the control w, Maurer states that the the optimal controller is given by $w(\cdot) = -\mathrm{sgn}(\phi(\cdot))$, which means the optimal control is dependent on the sign of the switching function which depends on the sign of the costate $p_{n+1}(\cdot)$.

The terms containing $\eta(\cdot)$ can be understood as slack terms, such that when the trajectory hits the boundary of the states, the function $\eta(\cdot)$ starts changing from 0, until the next switching time, when

$\eta(\cdot)$ reaches 0 again. The switching function is given by:

$$\phi(\cdot) = \mathcal{P}^T(t)\varphi(t)_0 = 0,$$
$$\dot{\phi}(\cdot) = -\gamma\mathcal{P}^T(\cdot)g_{x_{n+1}}(X(\cdot)) - \gamma S_{x_{n+1}}(x_{n+1}(\cdot))\eta(\cdot) = 0.$$

Solving for $\eta(\cdot)$ gives:

$$\eta(t) = \frac{-\mathcal{P}^T(t)g_{x_{n+1}}(X(t))}{S_{x_{n+1}}(x_{n+1}(t))} \tag{IV.10}$$

where $t \in \{[t_0, t_f] \mid x_{n+1}(t) \in \{0,1\}\}$. The constant γ disappears, meaning the weight put on the control does not matter, but the weight on the added cost function does matter since it appears in the derivative of $L_v(x_{n+1}(\cdot))$ with respect to x_{n+1} in $g_{x_{n+1}}(X(\cdot))$.

The system then follows a duality, changing between a boundary arc with $w(\cdot) = 0$ but $\eta(\cdot) \neq 0$, and an interior arc where the augmented state is in the interior of its range and the control is at a boundary. Thus the augmented state $x_{n+1}(\cdot)$ is either 0 or 1 most of the time, and whenever $\eta(\cdot) = 0$, the augmented state changes at a rate $\pm\gamma$ until it reaches the boundary of 1 or 0.

The difference between the AMEOCP and the MEOCP is that the weight, β, on the added cost explicitly affects the dwell time of the system through IV.10. A minimum dwell time constraint can be met by adjusting β such that the switching frequency of the system is below the threshold set by the constraint. Unfortunately, since the dwell time is dictated by the costates through $\eta(\cdot)$, a trial-and-error approach using numerical computation of the costate is required to select β. Further research is needed to formally characterize the relationship between the added cost and the minimum dwell time. The following section illustrates the AMEOCP using the two-tank system.

IV.3 Example

The two tank system is used in this example, with $\beta = 0.2$ being the weight on the added cost, and $\gamma = 10$ being the weight on the augmented state control. The states of this system are $x(\cdot) \in \mathbb{R}^2 \times [0,1]$ with control $w(\cdot) \in [-1,1]$ and the dynamics:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 1 + v_c(t) - \sqrt{x_1(t)} \\ \sqrt{x_1(t)} - \sqrt{x_2(t)} \\ \gamma w(t) \end{bmatrix}, \ \forall t \in [0, 20]. \tag{IV.11}$$

The cost function of the AMEOCP is:

$$J(x(\cdot), w(\cdot)) = \int_{t_0}^{t_f} \alpha(x_2(\tau) - 3)^2 + \beta(x_3(\tau) - x_3^2(\tau))d\tau \tag{IV.12}$$

The solution shown in Fig. IV.1, IV.2, IV.3, and IV.4 using GPOPS is similar to those shown before, but now the embedded control v_c identified with the third state x_3 and the new control is its derivative scaled by $\frac{1}{\gamma}$. For almost all of the time range, the state $x_3(\cdot)$ is at the boundary of the constraint set, which means the new control $w(\cdot)$ is near 0 (see Fig. IV.4). Due to the approximate nature of the numerical solution, there is some variation in the control to keep the system at the extrema of the augmented state. Whenever the system switches, the control jumps to its maximum positive/negative value to change the augmented state across its interior as quickly as possible, before dropping back to 0 when the system hits a boundary again. Theoretically, the weight γ on the control $w(\cdot)$ should not matter, but because GPOPS uses a collocation method, γ needs to be chosen such that a collocation point falls in the time interval where x_3 is in the interior of its range. Thus, if many collocation points are used, γ can be large so that the time interval of the change is small and the majority of the time, $x_3(t) \in \{0, 1\} \; \forall t \in [t_0, t_f]$. The value of 10 used was found by trial and error, but a solver built with this method in mind can be made to specifically choose points in the interior no matter how high the weight is, so the effect of the switching time can be minimized.

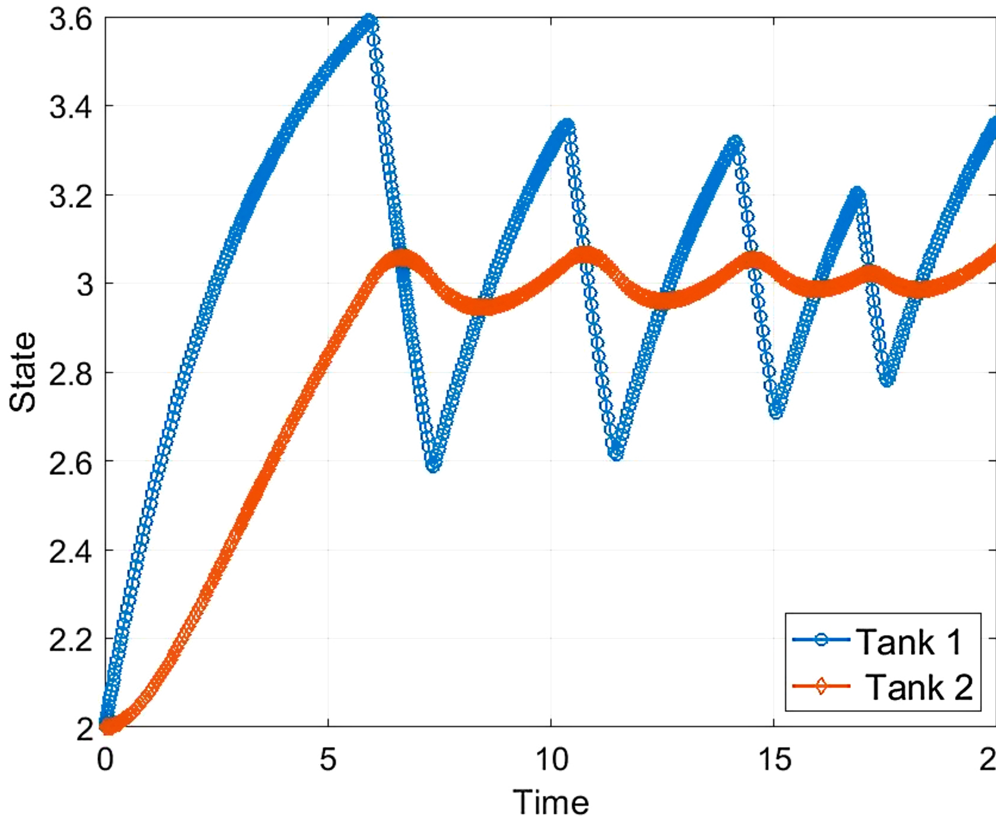

Figure IV.1: States 1 and 2 of the optimal solution to the two tank system with a weight of 0.5.

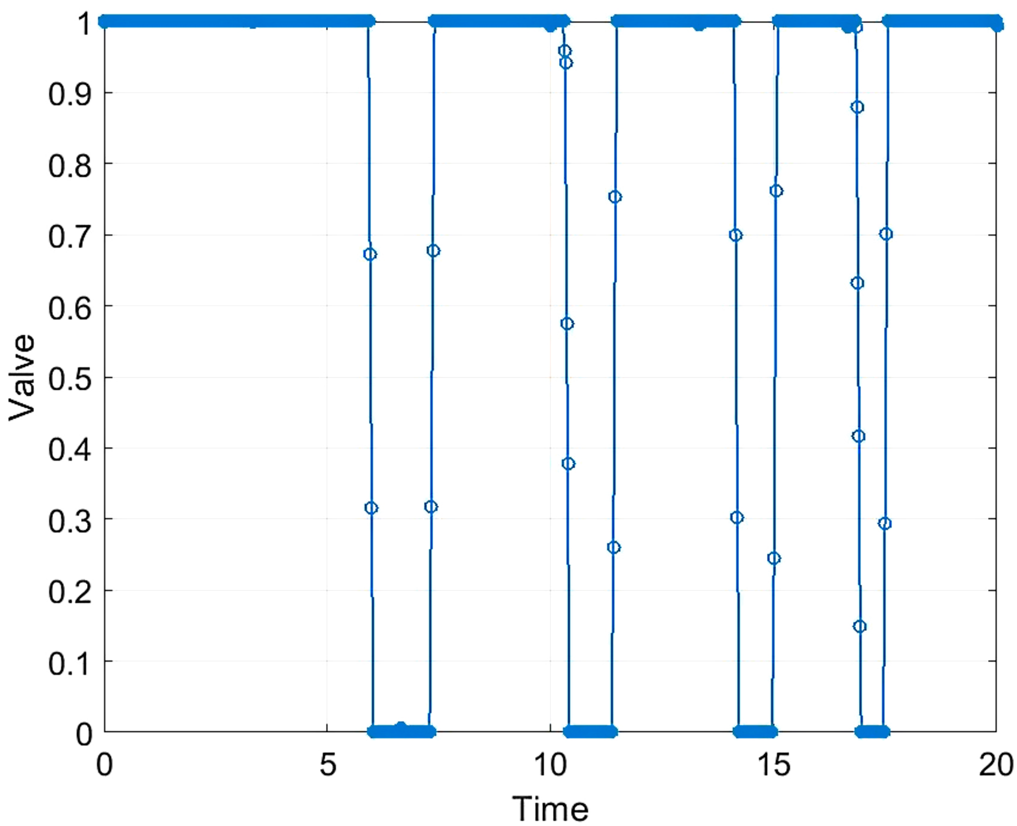

Figure IV.2: State 3 of the optimal solution to the two tank system with a weight of 0.5.

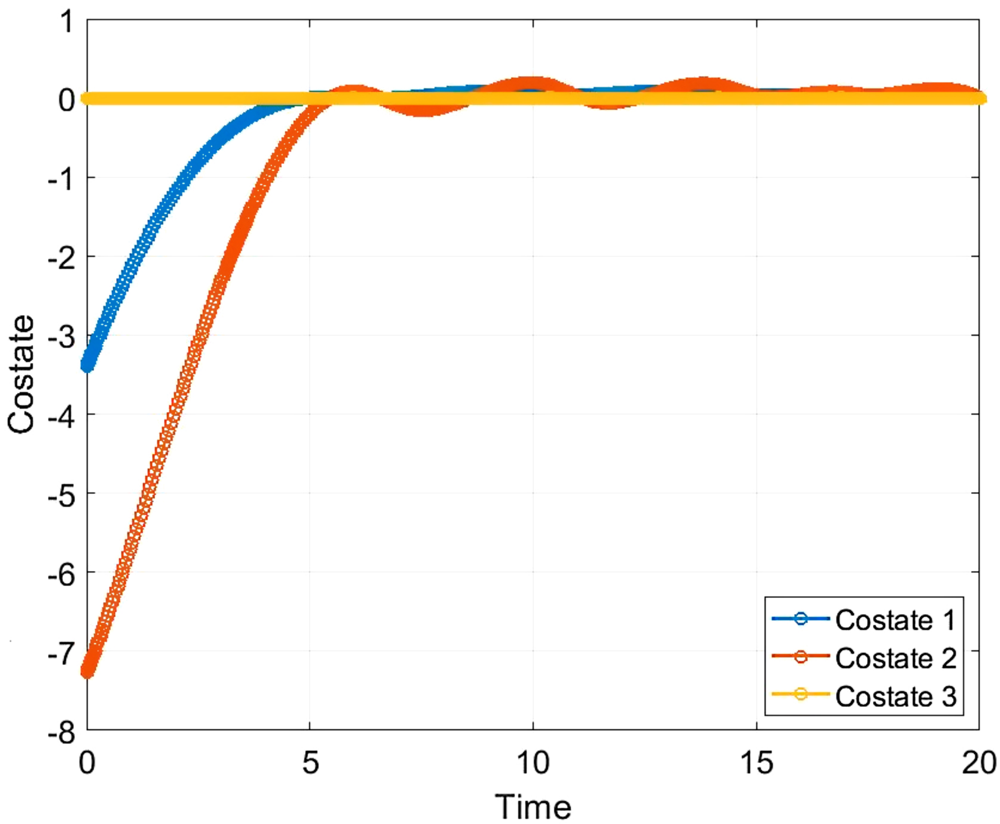

Figure IV.3: The control signal of the optimal solution of the two tank system with a weight of 0.5.

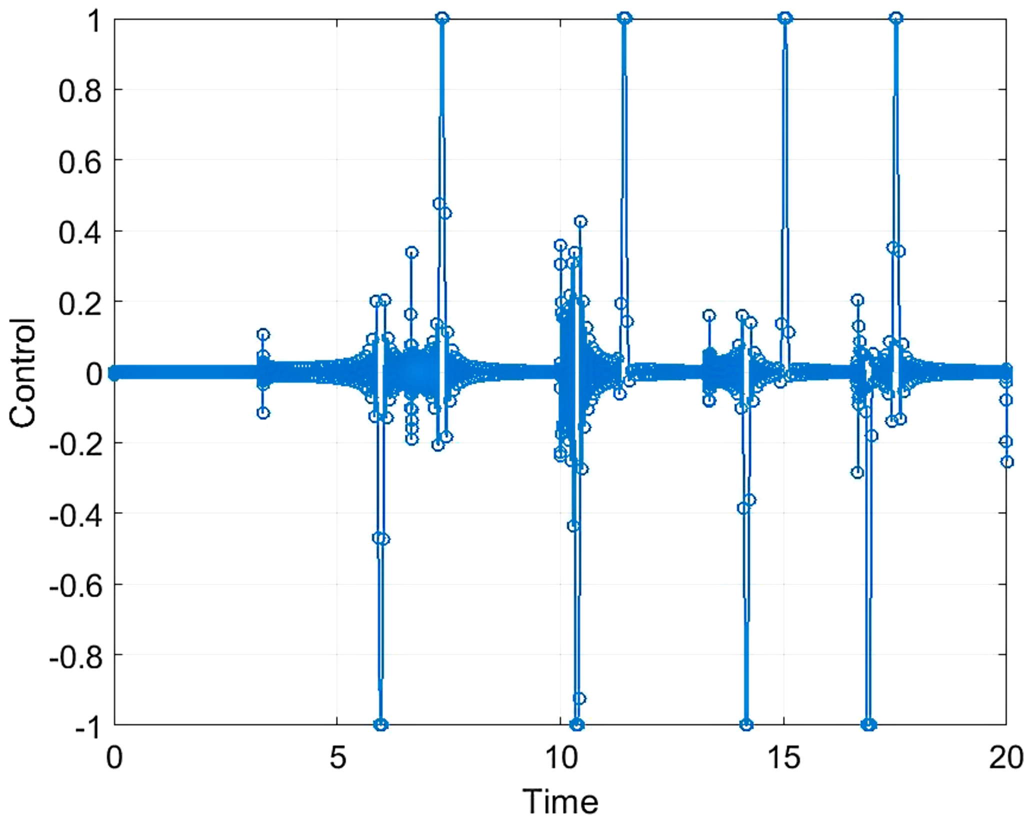

Figure IV.4: The control signal of the optimal solution of the two tank system with a weight of 0.5.

CHAPTER V

Conclusions

This thesis develops a new technique for solving switched optimal control problems with minimum dwell time constraints. The developed technique is inspired by the embedding process of Bengea and DeCarlo [20]. To ensure that solutions of the embedded optimal control problem can be implemented on the switched system, i.e. are bang-bang, a modified embedding process is developed and analyzed. The modification also yields a way to change the dwell time of the optimal bang-bang solution. The modification involves the introduction of an auxiliary cost that penalizes the embedded control to ensure that the optimal solution is pushed to the boundary of the control constraint set, and as a result, is implementable in the switched system. The dwell-time in this formulation is shown to be solver-dependent, and hence, unpredictable. To remove the solver dependence, the system dynamics are expressed in terms of a new state that includes the embedded control and a new control signal that is the derivative of the embedded control. The optimal solution of the resulting augmented problem is shown to possess the desired characteristics using Pontryagin's Minimum Principle for constrained optimal control problems in which the control appears linearly. The dwell-time is determined by the weight that multiplies the added cost and the solution is implementable in the original switched system.

Using the developed method, the dwell-time of the optimal solution can only be calculated using the optimal trajectory of the switching function, which involves computing the optimal trajectory of the states and costates. As a result, the research in this thesis does not yield a method to compute the auxiliary cost to achieve a given dwell-time. Thus, the added cost must be adjusted via trial and error until the solution satisfies the minimum dwell-time constraint, which requires numerical computation of the optimal states and costates for every auxiliary cost. While computationally taxing, the developed method is preferable to using the chattering method proposed in [33]. The chattering method relies on a bang-bang *approximation* of a singular solution of the embedded optimal control problem, and as such, the terminal constraints and the path constraints of the original switched optimal control problem may be violated by the bang-bang approximation. In contrast, the method developed in this thesis results in *feasible* bang-bang solutions.

V.1 Future Work

This research can be extended in several directions. The first and greatest question that needs to be answered is how to determine the minimum dwell time of a system with a given auxiliary weighting without having to resort to a trial and error process of calculating the optimal states and costates for each weighting value. Currently, the relationship between a dwell time constraint and $\eta(\cdot)$ is not adequately understood, and implementing a specific minimum dwell time constraint requires calculating arcs of $\eta(\cdot)$. The process requires much computation, but finding a direct relation would reduce the computation required extensively.

Another direction that should be investigated is the application of the developed method to systems with multiple discrete variables. The examples given and the proof are for a single discrete variable, but [20] includes extensions to multiple discrete variables, so this research should be extended as well. This research can also be extended by exploring how the continuous controls introduced in the original switched system affect the results of the proof by Maurer. It is possible to solve for these controls via another way before solving the switched system using the developed method. However, if the continuous controls appear linearly in the switched optimal control problem, they can be included directly in the developed method and the proof still holds.

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