On reduction of differential inclusions and Lyapunov stability

Rushikesh Kamalapurkar, Warren E. Dixon, and Andrew R. Teel

***Abstract*— In this paper, locally Lipschitz regular functions are utilized to identify and remove infeasible directions in differential inclusions. The resulting reduced differential inclusion is point-wise smaller (in the sense of set containment) than the original differential inclusion. The reduced inclusion is utilized to develop a generalized notion of a time derivative for locally Lipschitz candidate Lyapunov functions. The developed generalized derivative yields less conservative statements of Lyapunov stability results and invariance-like results for differential inclusions. Illustrative examples are included to demonstrate the utility of the developed stability theorems.**

I. Introduction

Differential inclusions can be used to model and analyze a large variety of practical systems. For example, systems that utilize discontinuous control architectures such as sliding mode control, multiple model and sparse neural network adaptive control, finite state machines, gain scheduling control, etc., can be modeled using differential inclusions. Differential inclusions can also be used to analyze robustness to bounded perturbations and modeling errors, to model physical phenomena such as coulomb friction and impact, and to model differential games (see, e.g., [1], [2]).

Asymptotic properties of trajectories of differential inclusions are typically analyzed using Lyapunov-like comparison functions. Nonsmooth candidate Lyapunov functions can be utilized to analyze trajectories of differential inclusions using several generalized notions of directional derivatives. Early results on stability of differential inclusions that utilize nonsmooth candidate Lyapunov functions are based on Dini directional derivatives (see, e.g., [3], [4]) and contingent derivatives (see, e.g., [5, Chapter 6]).

For locally Lipschitz regular candidate Lyapunov functions, stability results based on Clarke's notion of generalized directional derivatives have been developed in results such as [6]–[8]. In [6], Paden and Sastry utilize the Clarke gradient to develop a set-valued generalized derivative along with several Lyapunov-based stability theorems. In [7], Bacciotti and Ceragioli introduce another set-valued generalized derivative that results in sets that are smaller, pointwise, than those generated by the set-valued derivative in [6]; hence, the Lyapunov theorems in [7] are generally less conservative than their counterparts in [6]. The Lyapunov theorems by Ceragioli et al. have also been shown to be less conservative than those based on Dini and contingent derivatives provided locally Lipschitz regular candidate Lyapunov functions are employed (cf. [9, Proposition 7]).

In this paper, locally Lipschitz regular functions are utilized to identify and remove infeasible directions in differential inclusions to yield a pointwise smaller (in the sense of set containment) equivalent differential inclusion. Using the reduced differential inclusion, a novel generalization of the set-valued derivative concepts in [6] and [7] is introduced for locally Lipschitz Lyapunov functions. The developed technique yields less conservative statements of Lyapunov stability (cf. [3], [4], [6], [7], [10], [11]), the invariance principle (cf. [8], [12]–[14]), and invariance-like theorems (cf. [15], [16, Theorem 2.5]) for differential inclusions.

The paper is organized as follows. Section II introduces the concepts of Clarke-gradient-based set-valued derivatives from [6] and [7] and presents a typical Lyapunov stability result. In Section III, a technique based on locally Lipschitz regular functions is developed to identify the infeasible directions in a differential inclusion. Section III also develops a novel generalization of the notion of time-derivative with respect to a differential inclusion. Section IV states Lyapunov stability results that utilize the novel definition of the generalized time-derivative for autonomous and nonautonomous differential inclusions. Illustrative examples are also presented to demonstrate that the developed stability theory can be less conservative than results such as [6] and [7]. Section V summarizes the article and includes concluding remarks.

II. Set-valued derivatives

Let[1] $F : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \rightrightarrows \mathbb{R}^n$ be an upper semi-continuous [17, Definition 1.4.1] map with compact, nonempty, and convex values. Then, solutions to the differential inclusion

$$\dot{x} \in F(x, t) \tag{1}$$

exist over some interval $\mathcal{I} \subseteq \mathbb{R}_{\geq t_0}$ [1, p. 77]. To facilitate the discussion, let $\mathcal{D} \subseteq \mathbb{R}^n$ be open and connected, $\Omega \triangleq \mathcal{D} \times \mathcal{I}$, let $\mathcal{I}_{\mathcal{D}} \triangleq [t_0, T_{\mathcal{D}})$ where $T_{\mathcal{D}}$ denotes the first exit time of x from $\mathcal{D}$, i.e., $T_{\mathcal{D}} \triangleq \min\left(\sup \mathcal{I}, \inf\left\{t \in \mathcal{I} \mid x(t) \notin \mathcal{D}\right\}\right)$, where $\inf \emptyset$ is assumed to be ∞. Since $\mathcal{D}$ is open and $t \mapsto x(t)$ is locally absolutely continuous, provided $x(t_0) \in \mathcal{D}$, then $\mathcal{I}_{\mathcal{D}} \neq \emptyset$. The notion of a solution to (1) is defined in the following.

Rushikesh Kamalapurkar is with the School of Mechanical and Aerospace Engineering, Oklahoma State University, Stillwater, OK, USA. rushikesh.kamalapurkar@okstate.edu.

Warren E. Dixon is with the Department of Mechanical and Aerospace Engineering, University of Florida, Gainesville, FL, USA. wdixon@ufl.edu.

Andrew R. Teel is with the Department of Electrical and Computer Engineering, University of California, Santa Barbara, CA, USA. teel@ece.ucsb.edu.

[1] For $a \in \mathbb{R}$, the notation $\mathbb{R}_{\geq a}$ denotes the interval $[a, \infty)$ and the notation $\mathbb{R}_{>a}$ denotes the interval (a, ∞). The notation $F : A \rightrightarrows B$ is used to denote a set-valued map from A to the subsets of B.

Definition 1. *[1, p. 50] A locally absolutely continuous function $x : \mathcal{I} \to \mathbb{R}^n$ is called a solution to (1) if*

$$\dot{x}(t) \in F(x(t), t), \tag{2}$$

for almost all $t \in \mathcal{I}$.

The focus of this article is on the development of a less conservative Lyapunov method for analysis of differential inclusions using Clarke's notion of generalized directional derivatives [18, p. 39]. In [4] Paden and Sastry introduce the following generalized notion of the time-derivative of a locally Lipschitz and regular candidate Lyapunov function with respect to a differential inclusion.

Definition 2. *[4] For a locally Lipschitz and regular [18, Definition 2.3.4] function $V : \Omega \to \mathbb{R}$, the set-valued derivative of V with respect to (1) is defined as*[2]

$$\dot{\tilde{V}}^F(x, t) \triangleq \bigcap_{p \in \partial V(x,t)} p^T [F(x, t); \{1\}],$$

where the ∂V denotes the generalized gradient of V defined as (see also, [18, Theorem 2.5.1])

$$\partial V(x, t) \triangleq$$
$$\overline{\text{co}} \{\lim \nabla V(x_i, t_i) \mid (x_i, t_i) \to (x, t), (x_i, t_i) \notin \Omega_V\}, \tag{3}$$

where Ω_V is the set of measure zero where the gradient of V is not defined.

Lyapunov stability theorems developed using the set-valued derivative $\dot{\tilde{V}}^F$ exploit the property that every upper bound of the set $\dot{\tilde{V}}^F(x(t), t)$ is also an upper bound of $\dot{V}(x(t), t)$, for almost all t where $\dot{V}(x(t), t)$ exists. The aforementioned fact is a consequence of the following Lemma.

Lemma 1. *[4] Let $x : \mathcal{I} \to \mathbb{R}^n$ be a solution of (1) and let $V : \Omega \to \mathbb{R}$ be a locally Lipschitz and regular function. Then, for each solution of (1) such that $x(t_0) \in \mathcal{D}$, $\dot{V}(x(t), t)$ exists for almost all $t \in \mathcal{I}_\mathcal{D}$ and $\dot{V}(x(t), t) \in \dot{\tilde{V}}^F(x(t), t)$, for almost all $t \in \mathcal{I}_\mathcal{D}$.*

Proof. See [4, Theorem 3]. □

In [7], the notion of a set-valued derivative is further generalized via the following definition.

Definition 3. *[7] For a locally Lipschitz and regular function $V : \Omega \to \mathbb{R}$, the set-valued derivative of V with respect to (1) is defined as*

$$\dot{\overline{V}}^F(x, t) \triangleq$$
$$\left\{a \in \mathbb{R} \,|\, \exists q \in F(x, t) \mid p^T[q; 1] = a, \forall p \in \partial V(x, t)\right\}. \tag{4}$$

The set-valued derivative in Definition 3 results in less conservative statements of Lyapunov stability than Definition (2) since it is contained within the set-valued derivative in Definition (2), and as evidenced by [9, Example 6], the containment can be strict. The Lyapunov stability theorems developed in [7] exploit the property that Lemma 1 also holds for $\dot{\overline{V}}^F$ (see [7, Lemma 1]).

The following proposition is an example of a typical Lyapunov stability result for time-invariant differential inclusions that utilizes set-valued derivatives of the candidate Lyapunov function. The proposition combines [7, Theorem 2] and a specialization of [6, Theorem 3.1]. Before stating the proposition, the following notions of Lyapunov stability are introduced.

Definition 4. *The differential inclusion $\dot{x} \in F(x)$ is said to be*

1) *stable at $x = 0$, if $\forall \epsilon > 0$, $\exists \delta > 0$ such that all solutions to $\dot{x} \in F(x)$ satisfy $x(0) \in \mathrm{B}(0, \delta) \implies x(t) \in \mathrm{B}(0, \epsilon), \forall t \geq 0$.*
2) *asymptotically stable at $x = 0$ if it is stable at $x = 0$ and $\exists c > 0$ such that all solutions to $\dot{x} \in F(x)$ satisfy $x(0) \in \mathrm{B}(0, c) \implies \lim_{t \to \infty} \|x(t)\| = 0$.*
3) *globally asymptotically stable at $x = 0$ if it is stable at $x = 0$ and all solutions to $\dot{x} \in F(x)$ satisfy $\lim_{t \to \infty} \|x(t)\| = 0$.*

Proposition 1. *Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be an upper semi-continuous map with compact, nonempty, and convex values. If $V : \mathbb{R}^n \to \mathbb{R}$ is a positive definite, locally Lipschitz and regular function such that either*[3] $\max \dot{\overline{V}}^F(x) \leq 0$ *or* $\max \dot{\tilde{V}}^F(x) \leq 0$, $\forall x \in \mathbb{R}^n$, *then $\dot{x} \in F(x)$ is stable at $x = 0$.*

Proof. See [7, Theorem 2] and [6, Theorem 3.1]. □

The following section details the key contribution of this article, i.e., the observation that locally Lipschitz regular functions can be utilized to reduce differential inclusions pointwise to sets of feasible directions that are smaller than the corresponding sets in (1). A novel and less conservative generalization of the notion of a time-derivative with respect to a differential inclusion is also developed to yield less conservative statements of results such as Proposition 1.

III. Generalized time derivatives - reduced differential inclusions

By definition, $\dot{\overline{V}}^F \subseteq \dot{\tilde{V}}^F$, which implies $\max \dot{\overline{V}}^F(x) \leq \max \dot{\tilde{V}}^F(x)$. In some cases, $\dot{\overline{V}}^F$ can be a proper subset of $\dot{\tilde{V}}^F$ (see, e.g., [7, Example 1]) i.e., $\max \dot{\overline{V}}^F(x) < \max \dot{\tilde{V}}(x)$, which implies that Lyapunov theorems based on $\dot{\overline{V}}^F$ can be less conservative than those based on $\dot{\tilde{V}}^F$. A tighter bound on the evolution of V as x moves along

[2]If $a \in \mathbb{R}^m$ and $b \in \mathbb{R}^n$ then $[a; b]$ denotes the vector $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^{m+n}$. If $A \subseteq \mathbb{R}^m$, $B \subseteq \mathbb{R}^n$ then $[A; B]$ denotes the set $\left\{\begin{bmatrix} a \\ b \end{bmatrix} \mid a \in A, b \in B\right\}$.

[3]The definitions of $\dot{\overline{V}}^F$ and $\dot{\tilde{V}}^F$ translate to time-invariant systems as $\dot{\overline{V}}^F(x) = \left\{a \in \mathbb{R} \mid \exists q \in F(x) \mid p^T q = a, \forall p \in \partial V(x)\right\}$ and $\dot{\tilde{V}}^F(x) \triangleq \bigcap_{p \in \partial V(x,t)} p^T F(x)$, respectively and $\mathrm{B}(x, \epsilon) \subset \mathbb{R}^n$ denotes the open ball $\{y \in \mathbb{R}^n \mid \|y - x\| < \epsilon\}$.

an orbit of (1) can be obtained by examining the following alternative representation of $\max \dot{\overline{V}}^F$,[4]

$$\max \dot{\overline{V}}^F (x,t) = \min_{p \in \partial V(x,t)} \max_{q \in G_V^F(x,t)} p^T [q; 1], \quad (5)$$

where, for any locally Lipschitz and regular function $U : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \to \mathbb{R}$, and any upper semi-continuous map $H : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \rightrightarrows \mathbb{R}^n$, with compact, nonempty, and convex values, the reduction $G_U^H : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \rightrightarrows \mathbb{R}^n$ is defined as

$$G_U^H (x,t) \triangleq \left\{ q \in H(x,t) \mid \exists a \in \mathbb{R} \mid p^T [q; 1] = a, \forall p \in \partial U(x,t) \right\}. \quad (6)$$

Proposition 1 and (5) suggest that the only directions in F that affect the stability properties of solutions to (1) are those included in G_V^F. That is, the directions that, through the inner product, map the Clarke gradient of V into a singleton. The key observation in this paper is that the statement above remains true even if V is replaced with any arbitrary locally Lipschitz regular function U. The following proposition formalizes the aforementioned observation.

Proposition 2. *Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be an upper semi-continuous map with compact, nonempty, and convex values, let $V : \mathbb{R}^n \to \mathbb{R}$ be a positive definite, locally Lipschitz, and regular function and let $U : \mathbb{R}^n \to \mathbb{R}$ be any other locally Lipschitz and regular function. If*

$$\min_{p \in \partial V(x)} \max_{q \in G_U^F(x)} p^T q \leq 0, \forall x \in \mathbb{R}^n,$$

then $\dot{x} \in F(x)$ is stable at $x = 0$.

Proof. The proposition follows from the more general result stated in Theorem 2. ☐

Proposition 2 indicates that locally Lipschitz regular functions help discover the admissible directions in F. That is, from the point of view of Lyapunov stability, only the directions in G_U^F are relevant, where U can be different from the Lyapunov function V. If $U = V$, Proposition 2 reduces to Proposition 1.

To further generalize Proposition 2, the following concept of a reduced differential inclusion is introduced.

Definition 5. *Let $\{U_i\}_{i=1}^{\infty}$ be a collection of real-valued locally Lipschitz regular functions defined on Ω and let $\mathcal{U} \triangleq \{U_1, U_2, \cdots\}$. The set-valued map $\tilde{F}_U : \mathbb{R}^n \times \mathbb{R}_{\geq t_0} \rightrightarrows \mathbb{R}^n$, defined as*

$$\tilde{F}_{\mathcal{U}} (x,t) \triangleq F(x,t) \cap \left(\cap_{i=1}^{\infty} G_{U_i}^F (x,t) \right),$$

is called the $\mathcal{U}$-reduced differential inclusion for (1), where $G_{U_i}^F$ is introduced in (6).

The following theorem shows that the $\mathcal{U}$-reduced differential inclusion and the original differential inclusion are equivalent in the sense that the time-derivative of every solution to (1) is also contained within $\tilde{F}_{\mathcal{U}}$ in addition to F, for almost all t.

Theorem 1. *Let $\mathcal{U}$ be a collection of countably many real-valued locally Lipschitz regular functions defined on Ω and let $x : \mathcal{I} \to \mathbb{R}^n$ be a solution to (1). Then, $\dot{x}(t) \in \tilde{F}_{\mathcal{U}} (x(t), t)$ for almost all $t \in \mathcal{I}_{\mathcal{D}}$.*

Proof. The proof closely follows the proof of [7, Lemma 1]. Consider the set of times $\mathcal{T} \subseteq \mathcal{I}_{\mathcal{D}}$ where $\dot{x}(t)$ is defined, $\dot{U}_i(x(t), t)$ is defined $\forall i \geq 0$ and $\dot{x}(t) \in F(x(t), t)$. Since x is a solution of (1) and the functions U_i are locally Lipschitz, $\mu(\mathcal{I}_{\mathcal{D}} \setminus \mathcal{T}) = 0$, where μ denotes the Lebesgue measure on $\mathbb{R}$. The objective is to show that $\dot{x}(t) \in \tilde{F}_{\mathcal{U}} (x(t), t)$, not just $F(x(t), t)$.

Since each function U_i is locally Lipschitz, for $t \in \mathcal{T}$ the time derivative of U_i can be expressed as

$$\dot{U}_i(x(t), t) = \lim_{h \to 0} \frac{(U_i(x(t) + h\dot{x}(t), t+h) - U_i(x(t), t))}{h}.$$

Since each U_i is regular, for $i \geq 1$,

$$\begin{aligned} \dot{U}_i (x(t), t) &= U_{i+}' ([x(t); t], [\dot{x}(t); 1]), \\ &= U_i^o ([x(t); t], [\dot{x}(t); 1]), \\ &= \max_{p \in \partial U_i(x(t), t)} p^T [\dot{x}(t); 1], \\ \dot{U}_i (x(t), t) &= U_{i-}' ([x(t); t], [\dot{x}(t); 1]), \\ &= U_i^o ([x(t); t], [\dot{x}(t); 1]), \\ &= \min_{p \in \partial U_i(x(t), t)} p^T [\dot{x}(t); 1], \end{aligned}$$

where U_+' and U_-' denote the right and left directional derivatives and U^o denotes the Clarke-generalized derivative [18, p. 39] of U. Thus, $p^T [\dot{x}(t); 1] = \dot{U}_i (x(t), t), \forall p \in \partial U_i (x(t), t)$, which implies $\dot{x}(t) \in G_{U_i}^F (x(t), t)$ for each i. Therefore, $\dot{x}(t) \in \tilde{F}_{\mathcal{U}}(x(t), t)$, $\forall t \in \mathcal{T}$. Since $\mu(\mathcal{I}_{\mathcal{D}} \setminus \mathcal{T}) = 0$, $\dot{x}(t) \in \tilde{F}_{\mathcal{U}}(x(t), t)$ for almost all $t \in \mathcal{I}_{\mathcal{D}}$. ☐

The following example demonstrates the utility of Theorem 1.

Example 1. *Consider the differential inclusion $\dot{x} \in F(x)$, where $x \in \mathbb{R}$ and $F : \mathbb{R} \rightrightarrows \mathbb{R}$ is defined as*

$$F(x) \triangleq \begin{cases} 2 \operatorname{sgn}(x-1) & |x| \neq 1 \\ [-2, 5] & |x| = 1 \end{cases},$$

where $\operatorname{sgn}(x)$ denotes the sign of x. Let $U : \mathbb{R} \to \mathbb{R}$ be defined as

$$U(x) \triangleq \begin{cases} |x| & |x| \leq 1 \\ 2|x| - 1 & |x| > 1 \end{cases}.$$

Then, U is Lipschitz and since it is convex, it is also regular [18, Proposition 2.3.6]. Since U is Lipschitz, given a measure zero set $N \subset \mathbb{R}^n \times \mathbb{R}_{\geq t_0}$ and the set $N_U \subset \mathbb{R}^n \times \mathbb{R}_{\geq t_0}$ where U is not differentiable, the Clarke gradient of U can

[4]The minimization in (5) serves to maintain consistency of notation but is in fact redundant.

be computed using (3) as

$$\partial U(x) = \begin{cases} [1,2] & x = 1 \\ [-2,-1] & x = -1 \\ \{\operatorname{sgn}(x)\} & 0 < |x| < 1 \\ \{2\operatorname{sgn}(x)\} & |x| > 1 \\ [-1,1] & |x| = 0 \end{cases}.$$

The set G_U^F is then given by

$$G_U^F(x) = \begin{cases} \{0\} & |x| = 1 \\ \emptyset & |x| = 0 \\ F(x) & \text{otherwise} \end{cases}.$$

Since $G_U^F(x) \subset F(x), \forall x \in \mathbb{R}$, Theorem 1 can be invoked to conclude that every solution to $\dot{x} \in F(x)$ satisfies $\dot{x}(t) \in \tilde{F}_{\{U\}}(x(t)) = G_U^F(x(t))$ for almost all $t \in \mathbb{R}_{\geq t_0}$.

Proposition 2 and Theorem 1 suggest the following notion of a generalized time derivative of V with respect to (1), which also facilitates a unified treatment of Lyapunov stability theory using regular as well as nonregular candidate Lyapunov functions.

Definition 6. *The $\mathcal{U}-$generalized time derivative of a locally Lipschitz function $V : \Omega \to \mathbb{R}$ with respect to (1), denoted by $\overline{\dot{V}}^{F}_{\mathcal{U}}(x,t)$ is defined as*

$$\overline{\dot{V}}^{F}_{\mathcal{U}}(x,t) \triangleq \min_{p \in \partial V(x,t)} \max_{q \in \tilde{F}_{\mathcal{U}}(x,t)} p^T [q;1], \tag{7}$$

if V is regular, and

$$\overline{\dot{V}}^{F}_{\mathcal{U}}(x,t) \triangleq \max_{p \in \partial V(x,t)} \max_{q \in \tilde{F}_{\mathcal{U}}(x,t)} p^T [q;1], \tag{8}$$

if V is not regular. The $\mathcal{U}-$generalized time derivative is understood to be $-\infty$ when $\tilde{F}_{\mathcal{U}}(x,t)$ is empty.

The candidate Lyapunov function will be called a Lyapunov function if the following conditions are satisfied.

Definition 7. *If $V : \Omega \to \mathbb{R}$ is positive definite and if $\overline{\dot{V}}^{F}_{\mathcal{U}}(x,t) \leq 0, \forall (x,t) \in \Omega$ then V is called a $\mathcal{U}-$generalized Lyapunov function for (1).*

If $V \in \mathcal{U}$ then $\tilde{F}_{\mathcal{U}} \subseteq G_V^F$, and hence, $\overline{\dot{V}}^{F}_{\mathcal{U}}(x,t) \leq \max \overline{\dot{V}}^{F}(x,t), \forall (x,t) \in \Omega$. Thus, by judicious selection of the functions in $\mathcal{U}$, $\overline{\dot{V}}^{F}_{\mathcal{U}}(x,t)$ can be constructed to be a generalized time derivative of V with respect to (1) that is less conservative than the set-valued derivatives in [6] and [7]. Naturally, if $\mathcal{U} = \{V\}$ then $\overline{\dot{V}}^{F}_{\mathcal{U}} = \overline{\dot{V}}^{F}$.

IV. Lyapunov Stability Theory

This section develops relaxed Lyapunov-like stability theorems for differential inclusions based on the observations in Section III. The following corollary of Theorem 1 is stated to facilitate the analysis.

Corollary 1. *Let $V : \Omega \to \mathbb{R}$ be a locally Lipschitz function. Suppose there exists a function $W : \Omega \to \mathbb{R}$ such that*

$$\overline{\dot{V}}^{F}_{\mathcal{U}}(x,t) \leq W(x,t), \forall (x,t) \in \Omega. \tag{9}$$

Then, for each solution of (1) such that $x(t_0) \in \mathcal{D}$,[5]

$$\dot{V}(x(t),t) \in (\partial V(x(t),t))^T \left[\tilde{F}_{\mathcal{U}}(x,t);1\right],$$

for almost all $t \in \mathcal{I}_{\mathcal{D}}$. Furthermore,

$$\dot{V}(x(t),t) \leq W(x(t),t),$$

for almost all $t \in \mathcal{I}_{\mathcal{D}}$.

Proof. Let $x : \mathcal{I} \to \mathbb{R}^n$ be a solution of (1) such that $x(t_0) \in \mathcal{D}$. Consider a set of times $\mathcal{T} \subseteq \mathcal{I}_{\mathcal{D}}$ where $\dot{x}(t)$, $\dot{V}(x(t),t)$, and $\dot{U}_i(x(t),t)$ are defined $\forall i \geq 0$ and $\dot{x}(t) \in \tilde{F}_{\mathcal{U}}(x(t),t)$. Using Theorem 1 and the facts that x is absolutely continuous and V is locally Lipschitz, $\mu(\mathcal{I}_{\mathcal{D}} \setminus \mathcal{T}) = 0$.

If V is regular then similar arguments as the proof of Theorem 1 can be used to conclude that $p^T[\dot{x}(t);1] = \dot{V}(x(t),t), \forall p \in \partial V(x(t),t), \forall t \in \mathcal{T}$. Thus, (7), (9), and Theorem 1 imply that $\dot{V}(x(t),t) \in (\partial V(x(t),t))^T \left[\tilde{F}_{\mathcal{U}}(x,t);1\right]$ and $\dot{V}(x(t),t) \leq W(x(t),t)$, for almost all $t \in \mathcal{I}_{\mathcal{D}}$.

If V is not regular then [9, Proposition 4] (see also, [19, Theorem 2]) can be used to conclude that, for almost every $t \in \mathcal{I}_{\mathcal{D}}$, $\exists p_0 \in \partial V(x(t),t)$ such that $\dot{V}(x(t),t) = p_0^T[\dot{x}(t),1]$. Thus, (8), (9), and Theorem 1 imply that $\dot{V}(x(t),t) \in (\partial V(x(t),t))^T \left[\tilde{F}_{\mathcal{U}}(x,t);1\right]$ and $\dot{V}(x(t),t) \leq W(x(t),t)$ for almost all $t \in \mathcal{I}_{\mathcal{D}}$. □

A. Autonomous systems and the invariance principle

In this section, $\mathcal{U}-$generalized Lyapunov functions are utilized to formulate less conservative extensions to stability and invariance results for autonomous differential inclusions of the form

$$\dot{x} \in F(x), \tag{10}$$

where $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is an upper semi-continuous [17, Definition 1.4.1] map with compact, nonempty, and convex values. The following Lyapunov stability theorem is a consequence of Corollary 1.

Theorem 2. *If there exists a $\mathcal{U}-$generalized Lyapunov function $V : \mathcal{D} \to \mathbb{R}$ for (10), then (10) is stable at $x = 0$. If in addition $\overline{\dot{V}}^{F}_{\mathcal{U}}(x) \leq -W(x), \forall x \in \mathcal{D}$, for some negative definite function $W : \mathbb{R}^n \to \mathbb{R}$, then (10) is asymptotically stable at $x = 0$. Furthermore, if $\mathcal{D} = \mathbb{R}^n$ and if the sublevel sets $\{x \in \mathbb{R}^n \mid V(x) \leq c\}$ are compact for all $c \in \mathbb{R}_{\geq 0}$, then (10) is globally asymptotically stable at $x = 0$.*

Proof. See [20]. □

The following example presents a case where tests based on $\overline{\dot{V}}^{F}$ and $\dot{\tilde{V}}^{F}$ are inconclusive but Theorem 2 can be used to establish asymptotic stability.

[5]For $A, B \subset \mathbb{R}^n$, $A^T B$ denotes the set $\{a^T b \mid a \in A, b \in B\}$.

Example 2. *Let $H:\mathbb{R}\rightrightarrows\mathbb{R}$ be defined as*

$$H(y)\triangleq\begin{cases}\{0\} & |y|\neq 1\\ [-1,1] & |y|=1\end{cases}.$$

Let $F:\mathbb{R}^2\rightrightarrows\mathbb{R}^2$ be defined as[6]

$$F(x)\triangleq\begin{bmatrix}\{-x_1+x_2\}+H(x_2)\\ \{-x_1-x_2\}+H(x_1)\end{bmatrix}.$$

Consider the differential inclusion $\dot{x}\in F(x)$ and the candidate Lyapunov function $V:\mathbb{R}^2\to\mathbb{R}$ defined as $V(x)\triangleq\frac{1}{2}\|x\|_2^2$. Since $V\in\mathcal{C}^1(\mathbb{R}^2)$, the set-valued derivatives $\overline{\dot{V}}^F$ in [7] and $\dot{\tilde{V}}^F$ in [6] are bounded by[7]

$$\overline{\dot{V}}^F(x),\dot{\tilde{V}}^F(x)\leq\left\{-x_1^2-x_2^2\right\}+x_2H(x_1)+x_1H(x_2). \tag{11}$$

Since both $\dot{\tilde{V}}^F$ and $\overline{\dot{V}}^F$ cannot be shown to be negative semidefinite everywhere, the test in (11) is insufficient to draw conclusions regarding stability of 10.

Let $U:\mathbb{R}^2\to\mathbb{R}$ be defined as

$$\begin{aligned}U(x)&=\max((x_1-1),0)-\min((x_1+1),0)\\&\quad+\max((x_2-1),0)-\min((x_2+1),0).\end{aligned}$$

Then, U is Lipschitz and since it is convex, it is also regular [18, Proposition 2.3.6]. The Clarke gradient of U is given by,

$$\partial U(x)=\begin{cases}\{[\operatorname{sgn}1(x_1);\operatorname{sgn}1(x_2)]\} & |x_1|\neq 1,|x_2|\neq 1\\ [[0,\operatorname{sgn}(x_1)];\{\operatorname{sgn}1(x_2)\}] & |x_1|=1,|x_2|\neq 1\\ [\{\operatorname{sgn}1(x_1)\};[0,\operatorname{sgn}(x_2)]] & |x_1|\neq 1,|x_2|=1\\ [[0,\operatorname{sgn}(x_1)];[0,\operatorname{sgn}(x_2)]] & |x_1|=1,|x_2|=1\end{cases},$$

where

$$\operatorname{sgn}1(y)\triangleq\begin{cases}0 & -1<y<1\\ \operatorname{sgn}(y) & \text{otherwise}\end{cases},$$

and $[0,\operatorname{sgn}(y)]$ denotes the interval $[0,1]$ if $y>0$ and the interval $[-1,0]$ if $y<0$. In this case, the reduced inclusion $G_U^F$is given by

$$G_U^F(x)=\begin{cases}F(x) & |x_1|\neq 1,|x_2|\neq 1\\ \emptyset & \text{otherwise}\end{cases}.$$

Since $G_U^F(x)\subset F(x),\forall x\in\mathbb{R}^2$, then $\tilde{F}_{\{U\}}=G_U^F$. Since $V\in\mathcal{C}^1(\mathbb{R}^2)$, $\partial V(x)=\left\{\frac{\partial V}{\partial x}(x)\right\}$, and hence, the $\{U\}$−generalized time derivative of V with respect to $\dot{x}=F(x)$ is given by

$$\begin{aligned}\overline{\dot{V}}^F_{\{U\}}(x)&=\max_{q\in\tilde{F}_{\{U\}}(x)}\left(\frac{\partial V}{\partial x}(x)\right)^T q,\\&=\begin{cases}\begin{bmatrix}x_1 & x_2\end{bmatrix}\begin{bmatrix}-x_1+x_2\\ -x_1-x_2\end{bmatrix} & |x_1|\neq 1,|x_2|\neq 1\\ -\infty & \text{otherwise}\end{cases},\\&\leq -x_1^2-x_2^2.\end{aligned}$$

Global asymptotic stability of (10) at $x=0$ is then follows from Theorem 2.

Analogs of the Barbashin-Krasovskii-LaSalle invariance principle for autonomous differential inclusions appear in results such as [7], [12], [21]. Less conservative estimates of the limiting invariant set than those developed in [7], [12], [21] can be obtained by using locally Lipschitz regular functions to reduce the admissible directions in F. For example, the following theorem extends the invariance principle developed by Bacciotti and Ceragioli (see [7, Theorem 3]).

Theorem 3. *Let $c\in\mathbb{R}$ be a constant. Let $V:\mathcal{D}\to\mathbb{R}$ be a locally Lipschitz function such that $V(x)\geq c$ and $\overline{\dot{V}}^F_{\mathcal{U}}(x)\leq 0$, $\forall x\in\mathcal{D}$. Select $r>0$ such that $\overline{\mathrm{B}}(0,r)\subset\mathcal{D}$*[8] *and let $C_l\subset L_l$ be any bounded connected component of the sublevel set $L_l\triangleq\left\{x\in\overline{\mathrm{B}}(0,r):V(x)\leq l\right\}$. Let*

$$E\triangleq\left\{x\in\mathcal{D}\mid\overline{\dot{V}}^F_{\mathcal{U}}(x)=0\right\}$$

and let M be the largest weakly invariant [12, Definition 2.7] set in $\overline{E}\cap C_l$, where $\overline{E}$ denotes the closure of E. Then, every solution of (10) such that $x(t_0)\in C_l$ is complete and satisfies $\lim_{t\to\infty}\operatorname{dist}(x(t),M)=0$.

Proof. See [20]. ☐

Theorem 3 is often applied in the form of the following corollary.

Corollary 2. *Let $V:\mathbb{R}^n\to\mathbb{R}$ be a locally Lipschitz and positive definite function. Let the sublevel sets $\{x\in\mathbb{R}^n\mid V(x)\leq c\}$ be compact for all $c\in\mathbb{R}_{\geq 0}$. If $\overline{\dot{V}}^F_{\mathcal{U}}(x)\leq 0$, $\forall x\in\mathbb{R}^n$, and if for each $\mu>0$, no complete solution to (10) remains in the level set $\{x\in\mathbb{R}^n\mid V(x)=\mu\}$ then (10) is globally asymptotically stable at $x=0$.*

Proof. See [20]. ☐

The following example demonstrates the utility of the developed invariance principle.

Example 3. *Let $H:\mathbb{R}\rightrightarrows\mathbb{R}$ be defined as*

$$H(y)\triangleq\begin{cases}\{0\} & |y|\neq 1\\ \left[-\frac{1}{2},\frac{1}{2}\right] & |y|=1\end{cases},$$

and let $F:\mathbb{R}^2\rightrightarrows\mathbb{R}^2$ be defined as

$$F(x)=\begin{bmatrix}\{x_2\}+H(x_2)\\ \{-x_1-x_2\}+H(x_1)\end{bmatrix}, \tag{12}$$

and consider the differential inclusion $\dot{x}\in F(x)$. The candidate Lyapunov function $V:\mathbb{R}^2\to\mathbb{R}$ is defined as $V(x)\triangleq{}^1\!/_2\|x\|_2^2$. Since $V\in\mathcal{C}^1(\mathbb{R}^2)$, the set-valued derivatives $\overline{\dot{V}}^F$ in [7] and $\dot{\tilde{V}}^F$ in [6] are bounded by

$$\overline{\dot{V}}^F(x),\dot{\tilde{V}}^F(x)\leq\left\{-x_2^2\right\}+x_2H(x_1)+x_1H(x_2). \tag{13}$$

[6]For $A,B\subset\mathbb{R}^n$, $A\pm B$ denotes the set $\{a\pm b\in\mathbb{R}^n\mid a\in A,b\in B\}$.

[7]The notation $A\leq B$ for sets $A\subset\mathbb{R}^n$ and $B\subset\mathbb{R}^n$ implies $a\leq b$, $\forall a\in A$ and $\forall b\in B$.

[8]$\overline{\mathrm{B}}(x,\epsilon)\subset\mathbb{R}^n$ denotes the closed ball $\{y\in\mathbb{R}^n\mid\|y-x\|\leq\epsilon\}$.

That is, both $\dot{\tilde{V}}^F(x)$ and $\dot{\overline{V}}^F(x)$ are not negative semidefinite everywhere, and hence, the test in (13) is inconclusive.

Let U be defined as in Example 2. Then, the $\{U\}$ −reduced differential inclusion corresponding to F is given by

$$\tilde{F}_{\{U\}}(x) = \begin{cases} F(x) & |x_1| \neq 1, |x_2| \neq 1 \\ [\{0\}; \{-1\} + [-\frac{1}{2}, \frac{1}{2}]] & x_1 = 1, x_2 = 0 \\ [\{0\}; \{1\} + [-\frac{1}{2}, \frac{1}{2}]] & x_1 = -1, x_2 = 0 \\ \emptyset & \text{otherwise} \end{cases}.$$

The $\{U\}$ −generalized time derivative of V with respect to $\dot{x} = F(x)$ is then given by

$$\begin{aligned} \dot{\overline{V}}^F_{\{U\}}(x) &\triangleq \max_{q \in \tilde{F}_{\{U\}}(x)} \begin{bmatrix} x_1 & x_2 \end{bmatrix} q, \\ &= \begin{cases} \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} x_2 \\ -x_1 - x_2 \end{bmatrix} & |x_1| \neq 1, |x_2| \neq 1 \\ \max \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \{0\} \\ [\frac{3}{2}, -\frac{1}{2}] \end{bmatrix} & x_1 = 1, x_2 = 0 \\ \max \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} \{0\} \\ [\frac{1}{2}, \frac{3}{2}] \end{bmatrix} & x_1 = -1, x_2 = 0 \\ -\infty & \text{otherwise} \end{cases}, \\ &\leq -x_2^2, \end{aligned}$$

In this case, the set E in Theorem 3 is given by $E = \{x \in \mathbb{R}^2 \mid x_2 = 0\}$. Since the level sets L_l are bounded and connected $\forall l \in \mathbb{R}_{\geq 0}$ and the largest invariant set contained within $\overline{E} \cap L_l$ is $\{[0; 0]\}$ $\forall l \in \mathbb{R}_{\geq 0}$, Theorem 3 can be invoked to conclude that all solutions to (10) converge to the origin.

From Corollary 1, $\dot{V}(x(t), t) \leq -x_2^2(t)$ for almost all $t \in \mathbb{R}_{\geq 0}$; hence, given any $\mu > 0$, a trajectory of (10) can remain on the level set $\{x \in \mathbb{R}^n \mid V(x) = \mu\}$ if and only if $x_2(t) = 0$ and $x_1(t) = \pm\sqrt{2\mu}$, for all $t \in \mathbb{R}_{\geq 0}$. From Theorem 1, the state $[x_1; x_2]$ can remain constant at $[\pm\sqrt{2\mu}; 0]$ for all $t \in \mathbb{R}_{\geq 0}$ only if $[0; 0] \in F([\pm\sqrt{2\mu}; 0])$, which is not true for the inclusion in (12). Therefore, Corollary 2 can be invoked to conclude that the system is globally asymptotically stable at $x = 0$.

B. Nonautonomous systems and invariance-like results

In this section, a basic Lyapunov-based stability result is stated for nonautonomous differential inclusions. Furthermore, a partial stability result based on Barbalat's lemma is also stated.

Theorem 4. *Let $0 \in F(0, t), \forall t \in \mathbb{R}_{\geq t_0}$ and let $\Omega \triangleq \mathcal{D} \times \mathbb{R}_{\geq t_0}$. Let $V : \Omega \to \mathbb{R}$ be a locally Lipschitz positive definite function. Assume that there exist continuous positive definite functions $\overline{W}, \underline{W} : \mathcal{D} \to \mathbb{R}$ such that*

$$\underline{W}(x) \leq V(x, t) \leq \overline{W}(x), \quad \forall (x, t) \in \Omega, \tag{14}$$

and $\dot{\overline{V}}^F_{\mathcal{U}}(x, t) \leq 0$, $\forall x \in \mathcal{D}$ and for almost all $t \in \mathbb{R}_{\geq t_0}$. Then, the origin is a uniformly stable equilibrium point of (1). In addition, if there exists a continuous positive definite function $W : \mathcal{D} \to \mathbb{R}$ such that

$$\dot{\overline{V}}^F_{\mathcal{U}}(x, t) \leq -W(x), \tag{15}$$

$\forall x \in \mathcal{D}$ and for almost all $t \in \mathbb{R}_{\geq t_0}$, then the origin is a uniformly asymptotically stable equilibrium point of (1). Furthermore, if $\mathcal{D} = \mathbb{R}^n$ and if the sublevel sets $\{x \in \mathbb{R}^n \mid \underline{W}(x) \leq c\}$ are compact $\forall c \in \mathbb{R}_{\geq 0}$, then the origin is a globally uniformly asymptotically stable equilibrium point of (1).

Proof. See [20]. □

In the following example, tests based on $\dot{\overline{V}}^F$ and $\dot{\tilde{V}}^F$ are inconclusive, but Theorem 4 can be invoked to conclude global uniform asymptotic stability of the origin.

Example 4. *Let $H : \mathbb{R} \rightrightarrows \mathbb{R}$ be defined as in Example 3 and let $F : \mathbb{R}^2 \times \mathbb{R}_{\geq t_0} \rightrightarrows \mathbb{R}^2$ be defined as*

$$F(x, t) = \begin{bmatrix} \{-x_1 + x_2(1 + g(t))\} + H(x_2) \\ \{-x_1 - x_2\} + H(x_1) \end{bmatrix},$$

where $g \in \mathcal{C}^1(\mathbb{R}_{\geq t_0})$, $0 \leq g(t) \leq 1, \forall t \in \mathbb{R}_{\geq t_0}$ and $\dot{g}(t) \leq g(t), \forall t \in \mathbb{R}_{\geq t_0}$. Consider the differential inclusion $\dot{x} \in F(x, t)$ and the candidate Lyapunov function $V : \mathbb{R}^2 \to \mathbb{R}$ defined as $V(x, t) \triangleq x_1^2 + (1 + g(t)) x_2^2$. Then, $\|x\|_2^2 \leq V(x, t) \leq 2\|x\|_2^2, \forall (x, t) \in \mathbb{R}^2 \times \mathbb{R}_{\geq t_0}$. In this case, since $V \in \mathcal{C}^1(\mathbb{R}^2 \times \mathbb{R}_{\geq t_0})$, similar to [22, Example 4.20], the set-valued derivatives $\dot{\overline{V}}^F$ in [7] and $\dot{\tilde{V}}^F$ in [6] satisfy the bound

$$\begin{aligned} \dot{\overline{V}}^F(x, t), \dot{\tilde{V}}^F(x, t) \leq \{-2x_1^2 - 2x_2^2\} + 2x_1 H(x_2) \\ + 2x_2 h(t) H(x_1), \end{aligned} \tag{16}$$

where $h(t) \triangleq 1 + g(t)$ and the inequality $2 + 2g(t) - \dot{g}(t) \geq 2$ is utilized. Therefore, both $\dot{\tilde{V}}^F(x, t)$ and $\dot{\overline{V}}^F(x, t)$ cannot be shown to be negative semidefinite everywhere.

Let U be defined as in Example 2. Then, the $\{U\}$ −reduced differential inclusion corresponding to F is given by

$$\tilde{F}_{\{U\}}(x, t) = \begin{cases} F(x, t) & |x_1| \neq 1, |x_2| \neq 1 \\ \emptyset & \text{otherwise} \end{cases}.$$

The $\{U\}$ −generalized time derivative of V with respect to $\dot{x} = F(x, t)$ is then given by

$$\begin{aligned} \dot{\overline{V}}^F_{\{U\}}(x) &\triangleq \max_{q \in \tilde{F}_{\{U\}}(x, t)} \left(\frac{\partial V}{\partial (x, t)}(x, t) \right)^T [q; 1], \\ &= \begin{cases} \begin{bmatrix} 2x_1 \\ 2x_2 h(t) \\ \dot{g}(t) x_2^2 \end{bmatrix}^T \begin{bmatrix} -x_1 + x_2 h(t) \\ -x_1 - x_2 \\ 1 \end{bmatrix} & \begin{array}{l} |x_1| \neq 1, \\ |x_2| \neq 1 \end{array} \\ -\infty & \text{otherwise} \end{cases}, \\ &\leq -2\|x\|_2^2, \end{aligned}$$

where $\lambda_{\min}(A)$ denotes the minimum Eigenvalue of the matrix $A \in \mathbb{R}^{2 \times 2}$. Theorem 4 can then be invoked to conclude that the origin is a globally uniformly asymptotically stable equilibrium point of $\dot{x} \in F(x, t)$.

In applications such as adaptive control, Lyapunov methods commonly result in semidefinite Lyapunov functions (i.e., candidate Lyapunov functions with time derivatives bounded by a negative semidefinite function of the state). The following theorem establishes the fact that if the function W in (15) is positive semidefinite then $t \mapsto W(x(t))$ asymptotically decays to zero.

Theorem 5. *Let $\Omega \triangleq \mathcal{D} \times \mathbb{R}_{\geq t_0}$ and let $V : \Omega \to \mathbb{R}$ be a locally Lipschitz positive definite function that satisfies (14) and (15) where $W : \mathcal{D} \to \mathbb{R}$ is positive semidefinite. Select $r > 0$ such that $\overline{\mathrm{B}}(0,r) \subset \mathcal{D}$. If F is locally bounded, uniformly in t, over Ω,[9] then, every solution to (1) such that $x(t_0) \in \left\{x \in \overline{\mathrm{B}}(0,r) \mid \overline{W}(x) \leq c\right\}$, where $c \triangleq \min_{\|x\|_2 = r} \underline{W}(x)$, is complete, bounded, and satisfies $\lim_{t\to\infty} W(x(t)) = 0$.*

Proof. See [20]. □

In the following example $\dot{\overline{V}}^F$ and $\dot{\tilde{V}}^F$ do not have a negative semidefinite upper bound, but Theorem 5 can be invoked to conclude partial stability.

Example 5. *Let $H : \mathbb{R} \rightrightarrows \mathbb{R}$ be defined as in Example 3 and let $F : \mathbb{R}^2 \times \mathbb{R}_{\geq t_0} \rightrightarrows \mathbb{R}^2$ be defined as*

$$F(x,t) = \begin{bmatrix} \{x_2(1+g(t))\} + H(x_2) \\ \{-x_1 - x_2\} + H(x_1) \end{bmatrix},$$

where $g \in \mathcal{C}^1(\mathbb{R}_{\geq t_0})$, $0 \leq g(t) \leq 1, \forall t \in \mathbb{R}_{\geq t_0}$ and $\dot{g}(t) \leq g(t), \forall t \in \mathbb{R}_{\geq t_0}$. Consider the differential inclusion $\dot{x} \in F(x,t)$ and the candidate Lyapunov function $V : \mathbb{R}^2 \to \mathbb{R}$ defined as $V(x,t) \triangleq x_1^2 + (1+g(t))x_2^2$. Then, $\|x\|_2^2 \leq V(x,t) \leq 2\|x\|_2^2, \forall (x,t) \in \mathbb{R}^2 \times \mathbb{R}_{\geq t_0}$. In this case, since $V \in \mathcal{C}^1\left(\mathbb{R}^2 \times \mathbb{R}_{\geq t_0}\right)$, the set-valued derivatives $\dot{\overline{V}}^F$ in [7] and $\dot{\tilde{V}}^F$ in [6] are bounded by

$$\dot{\overline{V}}^F(x,t), \dot{\tilde{V}}^F(x,t) \leq \left\{-2x_2^2\right\} + 2x_2 h(t) H(x_1) + 2x_1 H(x_2), \quad (17)$$

where $h(t) \triangleq 1 + g(t)$ and the inequality $2 + 2g(t) - \dot{g}(t) \geq 2$ is utilized. Thus, both $\dot{\tilde{V}}^F$ and $\dot{\overline{V}}^F$ are not negative semidefinite everywhere.

Let U be defined as in Example 2. Then, the $\{U\}$ −reduced differential inclusion corresponding to F is given by

$$\tilde{F}_{\{U\}}(x,t) = \begin{cases} F(x,t) & |x_1| \neq 1, |x_2| \neq 1 \\ \left[\{0\}; \{-1\} + \left[-\frac{1}{2}, \frac{1}{2}\right]\right] & x_1 = 1, x_2 = 0 \\ \left[\{0\}; \{1\} + \left[-\frac{1}{2}, \frac{1}{2}\right]\right] & x_1 = -1, x_2 = 0 \\ \emptyset & \text{otherwise} \end{cases}.$$

The $\{U\}$ −generalized time derivative of V with respect to $\dot{x} = F(x,t)$ is then given by

$$\begin{aligned} \dot{\overline{V}}^F_{\{U\}}(x) &\triangleq \max_{q \in \tilde{F}_{\{U\}}(x,t)} \left(\frac{\partial V}{\partial (x,t)}(x,t)\right)^T [q;1], \\ &= \begin{cases} \begin{bmatrix} 2x_1 \\ 2x_2 h(t) \\ \dot{g}(t) x_2^2 \end{bmatrix}^T \begin{bmatrix} x_2 h(t) \\ -x_1 - x_2 \\ 1 \end{bmatrix} & \begin{matrix} |x_1| \neq 1, \\ |x_2| \neq 1 \end{matrix} \\ \max \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \{0\} \\ \left[\frac{3}{2}, -\frac{1}{2}\right] \\ 1 \end{bmatrix} & \begin{matrix} x_1 = 1, \\ x_2 = 0 \end{matrix} \\ \max \begin{bmatrix} -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \{0\} \\ \left[\frac{1}{2}, \frac{3}{2}\right] \\ 1 \end{bmatrix} & \begin{matrix} x_1 = -1, \\ x_2 = 0 \end{matrix} \\ -\infty & \text{otherwise} \end{cases}, \\ &\leq -2x_2^2, \end{aligned}$$

Theorem 5 can then be invoked to conclude that $t \mapsto x_1(t) \in \mathcal{L}_\infty(\mathbb{R}_{\geq t_0})$ and $\lim_{t\to\infty} x_2(t) = 0$.

V. Conclusion

This paper demonstrates that locally Lipschitz regular functions can identify unfeasible directions in differential inclusions. The unfeasible directions can then be removed to yield a point-wise smaller (in the sense of set containment) equivalent differential inclusion. The reduction process is utilized to develop a novel generalization of the set-valued derivative for locally Lipschitz candidate Lyapunov functions. Less conservative statements of Lyapunov stability and invariance-like results for differential inclusions are developed based on the reduction using locally Lipschitz regular functions.

The fact that arbitrary locally Lipschitz regular functions can be used to reduce the differential inclusion to a smaller set of admissible directions indicates that there may be a *smallest* set of admissible directions corresponding to each differential inclusion. Further research is needed to establish the existence of such a set and to find a representation of it that facilitates computation.

[9] A set valued map $F : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \rightrightarrows \mathbb{R}^n$ is locally bounded, uniformly in t, over Ω, if for every compact $K \subset \mathcal{D}$, there exists $M > 0$ such that $\forall (x,t,y)$ such that $(x,t) \in K \times \mathbb{R}_{\geq t_0}$, and $y \in F(x,t)$, $\|y\|_2 \leq M$.

References

[1] A. F. Filippov, *Differential Equations with Discontinuous Right-hand Sides*. Kluwer Academic Publishers, 1988.

[2] N. N. Krasovskii and A. I. Subbotin, *Game-Theoretical Control Problems*. New York: Springer-Verlag, 1988.

[3] E. Roxin, "Stability in general control systems," *J. Differ. Equ.*, vol. 1, no. 2, pp. 115 – 150, 1965.

[4] B. E. Paden and S. S. Sastry, "A calculus for computing Filippov's differential inclusion with application to the variable structure control of robot manipulators," *IEEE Trans. Circuits Syst.*, vol. 34, no. 1, pp. 73–82, Jan. 1987.

[5] J. P. Aubin and A. Cellina, *Differential Inclusions*. Springer, Berlin, 1984.

[6] D. Shevitz and B. Paden, "Lyapunov stability theory of nonsmooth systems," *IEEE Trans. Autom. Control*, vol. 39 no. 9, pp. 1910–1914, 1994.

[7] A. Bacciotti and F. Ceragioli, "Stability and stabilization of discontinuous systems and nonsmooth Lyapunov functions," *ESAIM Control Optim. Calc. Var.*, vol. 4, pp. 361–376, 1999.

[8] Q. Hui, W. Haddad, and S. Bhat, "Semistability, finite-time stability, differential inclusions, and discontinuous dynamical systems having a continuum of equilibria," *IEEE Trans. Autom. Control*, vol. 54, no. 10, pp. 2465–2470, Oct. 2009.

[9] F. M. Ceragioli, "Discontinuous ordinary differential equations and stabilization," Ph.D. dissertation, Universita di Firenze, Italy, 1999.

[10] A. N. Michel and K. Wang, *Qualitative Theory of Dynamical Systems, the Role of Stability Preserving Mappings*. New York: Marcel Dekker, 1995.

[11] E. Moulay and W. Perruquetti, "Finite time stability of differential inclusions," *IMA J. Math. Control Info.*, vol. 22, pp. 465–275, 2005.

[12] E. Ryan, "An integral invariance principle for differential inclusions with applications in adaptive control," *SIAM J. Control Optim.*, vol. 36, no. 3, pp. 960–980, 1998.

[13] H. Logemann and E. Ryan, "Asymptotic behaviour of nonlinear systems," *Amer. Math. Month.*, vol. 111, pp. 864–889, 2004.

[14] A. Bacciotti and L. Mazzi, "An invariance principle for nonlinear switched systems," *Syst. Control Lett.*, vol. 54, pp. 1109–1119, 2005.

[15] W. M. Haddad, V. Chellaboina, and S. G. Nersesov, *Impulsive and Hybrid Dynamical Systems*. Princeton Series in Applied Mathematics, 2006.

[16] N. Fischer, R. Kamalapurkar, and W. E. Dixon, "LaSalle-Yoshizawa corollaries for nonsmooth systems," *IEEE Trans. Autom. Control*, vol. 58, no. 9, pp. 2333–2338, Sep. 2013.

[17] J. P. Aubin and H. Frankowska, *Set-valued analysis*. Birkhäuser, 2008.

[18] F. H. Clarke, *Optimization and nonsmooth analysis*. SIAM, 1990.

[19] J. J. Moreau and M. Valadier, "A chain rule involving vector functions of bounded variation," *J. Funct. Anal.*, vol. 74, no. 2, pp. 333–345, 1987.

[20] R. Kamalapurkar, W. E. Dixon, and A. R. Teel, "On reduction of differential inclusions and Lyapunov stability," in preparation.

[21] J. Alvarez, Y. Orlov, and L. Acho, "An invariance principle for discontinuous dynamic systems with applications to a Coulomb friction oscillator," *ASME J. Dyn. Syst. Meas. Control*, vol. 74, pp. 190–198, 2000.

[22] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Upper Saddle River, NJ: Prentice Hall, 2002.