

On reduction of differential inclusions and Lyapunov stability

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1. INTRODUCTION

Differential inclusions can be used to model and analyze a large variety of practical systems. For example, systems that utilize discontinuous control architectures such as sliding mode control, multiple model and sparse neural network adaptive control, finite state machines, gain scheduling control, etc., are analyzed using the theory of differential inclusions. Differential inclusions are also used to analyze robustness to bounded perturbations and modeling errors, to model physical phenomena such as coulomb friction and impact, and to model differential games [1], [2].

Asymptotic properties of trajectories of differential inclusions are typically analyzed using Lyapunov-like comparison functions. Several generalized notions of the directional derivative are utilized to characterize the change in the value of a candidate Lyapunov function along the trajectories of a differential inclusion. Early results on stability of differential inclusions that utilize nonsmooth candidate Lyapunov functions are based on Dini directional derivatives [3], [4] and contingent derivatives [5, Chapter 6]. For locally Lipschitz, regular candidate Lyapunov functions, stability results based on Clarke's notion of generalized directional derivatives have been developed in results such as [6]–[8]. In [6], Shevitz and Paden utilize the Clarke gradient to develop a set-valued generalized derivative along with several Lyapunov-based stability theorems. In [7], Bacciotti and Ceragioli introduce another set-valued generalized derivative that results in sets that are pointwise smaller than those generated by the set-valued derivative in [6]; hence, the Lyapunov theorems in [7] are generally less conservative than their counterparts in [6]. The Lyapunov theorems developed by Bacciotti and Ceragioli have also been shown to be less conservative than those based on Dini and contingent derivatives, provided locally Lipschitz, regular candidate Lyapunov functions are employed (cf. [9, Prop. 7]).

In this paper, and in the preliminary work in [10], locally Lipschitz, regular functions are utilized to identify and remove infeasible directions from a set-valued map that defines a differential inclusion to yield a pointwise smaller

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(in the sense of set containment) set-valued map that defines an equivalent reduced differential inclusion. Using the reduced differential inclusion, a novel generalization of the set-valued derivatives in [6] and [7] is introduced for locally Lipschitz candidate Lyapunov functions. The developed technique yields less conservative statements of Lyapunov stability results (cf. [3], [4], [6], [7], [11], [12]), invariance results (cf. [8], [13]–[15]), invariance-like results (cf. [16, Thm. 2.5], [17]), and Matrosov results (cf. [18]–[22]) for differential inclusions.

The paper is organized as follows. Section 2 introduces the notation. Sections 3 and 4 review differential inclusions and Clarke-gradient-based set-valued derivatives from [6] and [7], respectively. In Section 5, locally Lipschitz, regular functions are used to identify the infeasible directions in a set-valued map that defines a differential inclusion. Section 6 develops a novel generalization of the notion of a derivative in the direction(s) of a set-valued map. Sections 7 and 8 develop stability theory for autonomous and nonautonomous differential inclusions, respectively, using the new generalized derivative. Illustrative examples where the developed stability theory is less conservative than results such as [6] and [7] are presented. Section 9 summarizes the article and includes concluding remarks.

2. NOTATION

The n -dimensional Euclidean space is denoted by \mathbb{R}^n , μ denotes the Lebesgue measure on \mathbb{R}^n , \mathcal{D} denotes an open and connected subset of \mathbb{R}^n , and $\Omega := \mathcal{D} \times [0, \infty)$. Elements of \mathbb{R}^n are interpreted as column vectors and $(\cdot)^T$ denotes the vector transpose operator. The set of positive integers excluding 0 is denoted by \mathbb{N} . For $a \in \mathbb{R}$, $\mathbb{R}_{\geq a}$ denotes the interval $[a, \infty)$ and $\mathbb{R}_{>a}$ denotes the interval (a, ∞) . A set-valued map from A to the subsets of B is denoted by $F : A \rightrightarrows B$. For a set A , the convex hull, the closed convex hull, the closure, the interior, and the boundary are denoted by $\text{co} A$, $\overline{\text{co}} A$, \bar{A} , $\overset{\circ}{A}$, and $\text{bd}(A)$, respectively. If $a \in \mathbb{R}^m$ and $b \in \mathbb{R}^n$ then $[a ; b]$ denotes the concatenated vector $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^{m+n}$. For $A \subseteq \mathbb{R}^m$, $B \subseteq \mathbb{R}^n$, the set $\{[a ; b] \mid a \in A, b \in B\}$ is denoted by $\begin{bmatrix} A \\ B \end{bmatrix}$ or $[A ; B]$. For $A, B \subseteq \mathbb{R}^n$, $A^T B$ denotes the set $\{a^T b \mid a \in A, b \in B\}$, $A \pm B$ denotes the set $\{a \pm b \in \mathbb{R}^n \mid a \in A, b \in B\}$, and $A(\geq) \leq B$ implies $\|a\|(\geq) \leq \|b\|$, $\forall a \in A$, and $\forall b \in B$. For $x \in \mathbb{R}^n$ and $r, l > 0$, the sets $\{y \in \mathbb{R}^n \mid \|x - y\| \leq r\}$, $\{y \in \mathbb{R}^n \mid \|x - y\| < r\}$, and $\{y \in \mathbb{R}^n \mid r \leq \|y\| \leq l\}$ are denoted by $\bar{B}(x, r)$, $B(x, r)$ and $D(r, l)$, respectively. If $a \in \mathbb{R}$ then $|a|$ denotes the absolute value and if A is a set then $|A|$ denotes its cardinality. For $A \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, $\text{dist}(x, A) := \inf_{y \in A} \|x - y\|$. Essentially bounded, n -times continuously differentiable, and locally Lipschitz functions with domain A and codomain B are denoted by $\mathcal{L}_\infty(A, B)$, $\mathcal{C}^n(A, B)$, and $\text{Lip}(A, B)$, respectively. The zero element of \mathbb{R}^n is denoted by 0_n , with the subscript n suppressed whenever clear from the context. The notation \dot{V} is reserved for the total derivative of V with respect to time.

3. DIFFERENTIAL INCLUSIONS

Let $F : \Omega \rightrightarrows \mathbb{R}^n$ be a set-valued map. Consider the differential inclusion

$$\dot{x} \in F(x, t). \quad (1)$$

A locally absolutely continuous function $x : \mathcal{I}_x \rightarrow \mathcal{D}$ is called a *solution* to (1), with *interval of existence* $\mathcal{I}_x = [t_0, T)$, for some $0 \leq t_0 < T \leq \infty$, if $\dot{x}(t) \in F(x(t), t)$, for almost all $t \in \mathcal{I}_x$ [1, p. 50]. A solution is called *complete* if $\mathcal{I}_x = \mathbb{R}_{\geq t_0}$ and *maximal* if it does not have a proper right extension¹ which is also a solution to (1). If a solution is maximal and if the set $\overline{\{x(t) \mid t \in \mathcal{I}_x\}}$ is compact, then the solution is called *precompact*. Similar to [23, Prop. 1], Zorn's lemma can be used to show that every solution to (1) admits a right extension that is also a maximal solution to (1). Let $\mathcal{S}(\mathcal{E})$ denote the set of all maximal solutions to (1) such that $(x(t_0), t_0) \in \mathcal{E} \subseteq \Omega$ (in the case of an autonomous system, $\mathcal{S}(A)$ denotes the set of all maximal solutions to $\dot{x} \in F(x)$ where $x(t_0) \in A \subset \mathbb{R}^n$). The discussion in this article concerns set-valued maps that define differential inclusions that admit local solutions.

Definition 3.1. Let $F : \Omega \rightrightarrows \mathbb{R}^n$ be a set-valued map and $\mathcal{E} \subseteq \Omega$. The differential inclusion (1) is said to *admit local solutions over \mathcal{E}* if for all $(y, t_0) \in \mathcal{E}$, there exists $T \in \mathbb{R}_{>t_0}$ and a locally absolutely continuous function $x : [t_0, T) \rightarrow \mathcal{D}$ such that $x(t_0) = y$ and $\dot{x}(t) \in F(x(t), t)$ for almost all $t \in [t_0, T)$. \triangle

Sufficient conditions for the existence of local solutions to differential inclusions can be found in [1, §7, Thm. 1] and [1, §7, Thm. 5]. To assert the existence of complete solutions, the following notions of invariance are utilized in this article.

Definition 3.2. A set $A \subseteq \mathcal{D}$ is called *weakly forward invariant* with respect to (1) if $\forall x_0 \in A, \exists x(\cdot) \in \mathcal{S}(\{x_0\} \times \mathbb{R}_{\geq 0})$ such that $x(t) \in A, \forall t \in \mathcal{I}_x$. It is called *strongly forward invariant* with respect to (1) if every $x(\cdot) \in \mathcal{S}(A \times \mathbb{R}_{\geq 0})$ satisfies $x(t) \in A, \forall t \in \mathcal{I}_x$. \triangle

Forward invariance of a set $A \subseteq \mathcal{D}$ in the sense of Def. 3.2 does not imply completeness of any $x(\cdot) \in \mathcal{S}(A \times \mathbb{R}_{\geq 0})$ since $x(\cdot)$ can exit \mathcal{D} in finite time, resulting in a finite interval of existence \mathcal{I}_x . However, the following Lemma, which is a slight generalization of [23, Prop. 2], implies that under general conditions on F , if A is also compact then $\mathcal{S}(A \times \mathbb{R}_{\geq 0})$ contains complete solutions, and under strong forward invariance of A , all solutions in $\mathcal{S}(A \times \mathbb{R}_{\geq 0})$ are complete.

Lemma 3.3. Let $F : \Omega \rightrightarrows \mathbb{R}^n$ be a set-valued map such that (1) admits local solutions over Ω . Let $x(\cdot)$ be a maximal solution to (1) such that $\overline{\{x(t) \mid t \in \mathcal{I}_x\}} \subset \mathcal{D}$. If the set $\cup_{t \in \mathcal{J}} F(x(t), t)$ is bounded for every subinterval $\mathcal{J} \subseteq \mathcal{I}_x$ of finite length, then $x(\cdot)$ is complete.

Proof. For the sake of contradiction, assume that the interval of existence, \mathcal{I}_x , is finite. That is, $\mathcal{I}_x = [t_0, T)$ for some $t_0 < T < \infty$. Boundedness of the set $\cup_{t \in [t_0, T)} F(x(t), t)$ implies that $\dot{x}(\cdot) \in \mathcal{L}_\infty([t_0, T), \mathbb{R}^n)$. Since $x(\cdot)$ is locally absolutely continuous on $[t_0, T)$, it can be concluded that $\forall t_1, t_2 \in [t_0, T), \|x(t_2) - x(t_1)\|_2 = \left\| \int_{t_1}^{t_2} \dot{x}(\tau) d\tau \right\|_2$. Furthermore, $\dot{x}(\cdot) \in \mathcal{L}_\infty([t_0, T), \mathbb{R}^n)$ implies that $\left\| \int_{t_1}^{t_2} \dot{x}(\tau) d\tau \right\|_2 \leq \int_{t_1}^{t_2} M d\tau$, where M is a positive constant. Thus, $\|x(t_2) - x(t_1)\|_2 \leq M|t_2 - t_1|$, and hence, $x(\cdot)$ is uniformly continuous on $[t_0, T)$. Therefore, $x(\cdot)$ admits a continuous extension $x' : [t_0, T] \rightarrow \mathbb{R}^n$ [24, Chapter 4, Exercise 13]. Since $x'(\cdot)$ is

¹A solution $y : [t_0, T_y) \rightarrow \mathbb{R}^n$ to (1) is a (*proper*) *right extension* of a solution $x : [t_0, T_x) \rightarrow \mathbb{R}^n$ to (1) if $T_y (>) \geq T_x$ and $y(t) = x(t), \forall t \in [t_0, T_x)$.

continuous, \mathcal{D} is open, and $\overline{\{x(t) \mid t \in [t_0, T]\}} \subset \mathcal{D}$, it is clear that $x'(T) \in \mathcal{D}$. Since (1) admits local solutions over Ω , $x'(\cdot)$ can be extended into a solution to (1) on the interval $[t_0, T')$ for some $T' > T$, which contradicts the maximality of $x(\cdot)$. Hence, $x(\cdot)$ is complete. \square

Remark 3.4. The hypothesis of Lemma 3.3, that the set $\cup_{t \in \mathcal{J}} F(x(t), t)$ needs to be bounded for every subinterval $\mathcal{J} \subseteq \mathcal{I}_x$ of finite length, is met if, e.g., $(x, t) \mapsto F(x, t)$ is locally bounded over Ω and $x(\cdot)$ is precompact (cf. [25, Prop. 5.15]).

The following section presents a summary of the relevant Lyapunov methods that utilize Clarke's notion of generalized directional derivatives and gradients [26, p. 39] for the analysis of differential inclusions.

4. SET-VALUED DERIVATIVES

Clarke gradients are utilized in [6] by Shevitz and Paden to introduce the following set-valued derivative of a locally Lipschitz, positive definite (i.e., locally positive definite in the sense of [27, Sec. 5.2, Def. 3] at (x, t) , for all (x, t) in its domain) candidate Lyapunov function that is regular (i.e., regular at (x, t) , in the sense of [26, Def. 2.3.4], for all (x, t) in its domain).

Definition 4.1. [6] Given a regular function $V \in \text{Lip}(\Omega, \mathbb{R})$, and a set-valued map $F : \Omega \rightrightarrows \mathbb{R}^n$, the *set-valued derivative of V in the direction(s) F* is defined as

$$\dot{V}(x, t) := \bigcap_{p \in \partial V(x, t)} p^T \begin{bmatrix} F(x, t) \\ \{1\} \end{bmatrix}, \forall (x, t) \in \Omega,$$

where ∂V denotes the *Clarke gradient* of V , defined as (see also, [26, Thm. 2.5.1])

$$\partial V(x, t) := \overline{\text{co}} \{ \lim \nabla V(x_i, t_i) \mid (x_i, t_i) \rightarrow (x, t), (x_i, t_i) \in \Omega \setminus (\Omega_V \cup S) \}, \forall (x, t) \in \Omega, \quad (2)$$

where Ω_V is the set of Lebesgue measure zero where the gradient ∇V of V is not defined and $S \subset \Omega$ is any other set of Lebesgue measure zero. \triangle

Lyapunov stability theorems developed using the set-valued derivative \dot{V} exploit the property that every upper bound of the set $\dot{V}(x(t), t)$ is also an upper bound of $\dot{V}(x(t), t)$, for almost all t where $\dot{V}(x(t), t)$ exists. The aforementioned fact is a consequence of the following proposition.

Proposition 4.2. [6] *Let $x : \mathcal{I}_x \rightarrow \mathcal{D}$ be a solution to (1). If $V \in \text{Lip}(\Omega, \mathbb{R})$ is a regular function, then $\dot{V}(x(t), t)$ exists for almost all $t \in \mathcal{I}_x$ and $\dot{V}(x(t), t) \in \dot{V}(x(t), t)$, for almost all $t \in \mathcal{I}_x$.*

Proof. See [6, Thm. 2.2]. \square

In [7], the notion of a set-valued derivative is further generalized via the following definition.

Definition 4.3. [7] For a regular function $V \in \text{Lip}(\Omega, \mathbb{R})$ and a set-valued map $F : \Omega \rightrightarrows \mathbb{R}^n$, the *set-valued derivative of V in the direction(s) F* is defined as

$$\dot{V}(x, t) := \{ a \in \mathbb{R} \mid \exists q \in F(x, t) \mid p^T [q; 1] = a, \forall p \in \partial V(x, t) \}, \forall (x, t) \in \Omega. \quad \triangle$$

The set-valued derivative in Def. 4.3 results in less conservative statements of Lyapunov stability than Def. 4.1 since it is contained within the set-valued derivative in Def. 4.1 and, as evidenced by [7, Example 1], the containment can be strict. The Lyapunov stability theorems developed in [7] exploit the property that Prop. 4.2 also holds for $\dot{\bar{V}}$ (see [7, Lemma 1]).

In the following, notions of Lyapunov stability for differential inclusions are introduced.²

Definition 4.4. The differential inclusion $\dot{x} \in F(x)$ is said to be (strongly)

- (a) *stable* at $x = 0$ if $\forall \epsilon > 0 \exists \delta > 0$ such that every $x(\cdot) \in \mathcal{S}(\bar{B}(0, \delta))$ is complete and satisfies $x(t) \in \bar{B}(0, \epsilon)$, $\forall t \geq 0$.
- (b) *asymptotically stable* at $x = 0$ if it is stable at $x = 0$ and $\exists c > 0$ such that every $x(\cdot) \in \mathcal{S}(\bar{B}(0, c))$ is complete and satisfies $\lim_{t \rightarrow \infty} \|x(t)\| = 0$.
- (c) *globally asymptotically stable* at $x = 0$ if it is stable at $x = 0$ and every $x(\cdot) \in \mathcal{S}(\mathbb{R}^n)$ is complete and satisfies $\lim_{t \rightarrow \infty} \|x(t)\| = 0$. △

The following proposition is an example of a typical Lyapunov stability result for time-invariant differential inclusions that utilizes set-valued derivatives of the candidate Lyapunov function. The proposition combines [7, Thm. 2] and a specialization of [6, Thm. 3.1].

Proposition 4.5. Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be an upper semi-continuous map with compact, nonempty, and convex values. If $V \in \text{Lip}(\mathbb{R}^n, \mathbb{R})$ is a positive definite and regular function such that either³ $\max \dot{\bar{V}}(x) \leq 0$ or $\max \dot{\hat{V}}(x) \leq 0$, $\forall x \in \mathbb{R}^n$, then $\dot{x} \in F(x)$ is stable at $x = 0$.

Proof. See [7, Thm. 2] and [6, Thm. 3.1]. □

Inspired by [6] and [7], the following section presents a novel notion of reduced differential inclusions that results in statements of Lyapunov theorems such as Prop. 4.5. that are less conservative than those available in the literature.

5. REDUCED DIFFERENTIAL INCLUSIONS

By definition, $\dot{\bar{V}}(x, t) \subseteq \dot{\hat{V}}(x, t)$, $\forall (x, t) \in \Omega$, which, assuming compact values, implies $\max \dot{\bar{V}}(x, t) \leq \max \dot{\hat{V}}(x, t)$, $\forall (x, t) \in \Omega$. In some cases, $\max \dot{\bar{V}}$ can be strictly smaller than $\max \dot{\hat{V}}$ and Lyapunov theorems based on $\dot{\bar{V}}$ can be less conservative than those based on $\dot{\hat{V}}$ [7, Example 1]. A tighter bound on the evolution of V along an orbit of (1) can be obtained by examining the following equivalent representation of $\max \dot{\bar{V}}$:⁴

$$\max \dot{\bar{V}}(x, t) = \min_{p \in \partial V(x, t)} \max_{q \in G_V^p(x, t)} p^T [q; 1], \quad (3)$$

²While the results in this paper are stated in terms of stability at the origin, they extend in a straightforward manner to stability of arbitrary compact sets.

³The definitions of $\dot{\bar{V}}$ and $\dot{\hat{V}}$ translate to time-invariant systems as $\dot{\bar{V}}^F(x) = \{a \in \mathbb{R} \mid \exists q \in F(x) \mid p^T q = a, \forall p \in \partial V(x)\}$ and $\dot{\hat{V}}(x) := \bigcap_{p \in \partial V(x)} p^T F(x)$, respectively.

⁴The minimization in (3) serves to maintain consistency of notation, but is in fact, redundant.

where, for any regular function $U \in \text{Lip}(\Omega, \mathbb{R})$, and any set-valued map $H : \Omega \rightrightarrows \mathbb{R}^n$, the reduction $G_U^H : \Omega \rightrightarrows \mathbb{R}^n$ is defined as

$$G_U^H(x, t) := \{q \in H(x, t) \mid \exists a \in \mathbb{R} \mid p^\top [q; 1] = a, \forall p \in \partial U(x, t)\}, \forall (x, t) \in \Omega. \quad (4)$$

The representation in (3), along with Prop. 4.5, suggest that the only directions in F that affect the stability properties of solutions to (1) are those included in G_V^F , that is, the directions that map the Clarke gradient of V to a singleton. The key observation in this paper is that *the statement above remains true even when V is replaced by any arbitrary locally Lipschitz, regular function U* . The following proposition formalizes the aforementioned observation. For clarity, the proposition is stated here for autonomous differential inclusions. The analysis of nonautonomous differential inclusions is deferred to Thm. 8.2.

Proposition 5.1. *Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a locally bounded map with compact values such that $\dot{x} \in F(x)$ admits local solutions over \mathbb{R}^n . Let $V \in \text{Lip}(\mathbb{R}^n, \mathbb{R})$ be a positive definite and regular function and let $U \in \text{Lip}(\mathbb{R}^n, \mathbb{R})$ be any other regular function. If*

$$\min_{p \in \partial V(x)} \max_{q \in G_U^F(x)} p^\top q \leq 0, \quad \forall x \in \mathbb{R}^n,$$

then $\dot{x} \in F(x)$ is stable at $x = 0$.

Proof. The proposition follows from the more general result stated in Thm. 7.1 . □

Prop. 5.1 indicates that locally Lipschitz, regular functions help discover the admissible directions in F . That is, from the point of view of Lyapunov stability, only the directions in G_U^F are relevant, where U can be an arbitrary locally Lipschitz, regular function, possibly different from the candidate Lyapunov function V . If $U = V$, Prop. 5.1 reduces to Prop. 4.5.

In fact, the differential inclusion $\dot{x} \in G_U^F(x, t)$ is, in a sense, equivalent to the differential inclusion $\dot{x} \in F(x, t)$. To make the equivalence precise, the following definition of a reduced set-valued map is introduced.

Definition 5.2. Let $F : \Omega \rightrightarrows \mathbb{R}^n$ be a set-valued map and $\mathcal{U} := \{U_i\}_{i \in \mathcal{N}} \subset \text{Lip}(\Omega, \mathbb{R})$ be a countable collection of regular functions, indexed over $\mathcal{N} \subseteq \mathbb{N}$. The set-valued map $\tilde{F}_{\mathcal{U}} : \Omega \rightrightarrows \mathbb{R}^n$, defined as

$$\tilde{F}_{\mathcal{U}}(x, t) := \bigcap_{i \in \mathcal{N}} G_{U_i}^F(x, t) = \bigcap_{i \in \mathcal{N}} \{q \in F(x, t) \mid \exists a \in \mathbb{R} \mid p^\top [q; 1] = a, \forall p \in \partial U_i(x, t)\}, \forall (x, t) \in \Omega,$$

is called the \mathcal{U} -reduced set-valued map for F and the differential inclusion $\dot{x} \in \tilde{F}_{\mathcal{U}}(x, t)$ is called the \mathcal{U} -reduced differential inclusion for (1). If \mathcal{U} is empty, then $\tilde{F}_{\mathcal{U}} := F$. △

In other words, the \mathcal{U} -reduced set-valued map collects all directions q in F that, through the inner product $p^\top [q; 1]$, map the Clarke gradient of all functions in \mathcal{U} to a singleton. The following theorem demonstrates the key utility of the reduction in Def. 5.2, i.e., the reduced differential inclusion is found to be sufficient to characterize the solutions to (1).

Theorem 5.3. *If $x : \mathcal{I}_x \rightarrow \mathcal{D}$ is a solution to (1), then $\dot{x}(t) \in \tilde{F}_{\mathcal{U}}(x(t), t)$ for almost all $t \in \mathcal{I}_x$.*

Proof. The theorem can be proved using techniques similar to [7, Lemma 1]. \square

Although not directly related to the current discussion, it is worth mentioning that Thm. 5.3 also expands the class of differential inclusions that admit solutions, as detailed in the following corollary.

Corollary 5.4. *A differential inclusion $\dot{x} \in G(x, t)$, with $G : \Omega \rightrightarrows \mathbb{R}^n$, admits local solutions over $\mathcal{E} \subseteq \Omega$ if there exists: a set-valued map, $F : \Omega \rightrightarrows \mathbb{R}^n$, such that (1) admits local solutions over \mathcal{E} ; and a countable collection, $\mathcal{U} \subset \text{Lip}(\Omega, \mathbb{R})$, of regular functions, such that G is the \mathcal{U} -reduced set-valued map for F .*

The following example illustrates the utility of Thm. 5.3.

Example 5.5. Consider the differential inclusion in (1), where $x \in \mathbb{R}$, and $F : \mathbb{R} \times \mathbb{R}_{\geq 0} \rightrightarrows \mathbb{R}$ is defined as

$$F(x, t) := \begin{cases} 2 \operatorname{sgn}(x - 1) & |x| \neq 1, \\ [-2, 5] & |x| = 1, \end{cases}$$

where $\operatorname{sgn}(x)$ denotes the sign of x . The function $U : \mathbb{R} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$, defined as

$$U(x, t) := \begin{cases} |x| & |x| \leq 1, \\ 2|x| - 1 & |x| > 1, \end{cases}$$

satisfies $U \in \text{Lip}(\mathbb{R} \times \mathbb{R}_{\geq 0}, \mathbb{R})$. In addition, since U is convex, it is also regular [26, Prop. 2.3.6]. The Clarke gradient of U is given by

$$\partial U(x, t) = \begin{cases} [[1, 2] ; \{0\}] & x = 1, \\ [[-2, -1] ; \{0\}] & x = -1, \\ [\{\operatorname{sgn}(x)\} ; \{0\}] & 0 < |x| < 1, \\ [\{2 \operatorname{sgn}(x)\} ; \{0\}] & |x| > 1, \\ [[-1, 1] ; \{0\}] & x = 0. \end{cases}$$

The set G_U^F is then given by

$$G_U^F(x, t) = \begin{cases} \{0\} & |x| = 1, \\ \emptyset & x = 0, \\ F(x, t) & \text{otherwise.} \end{cases}$$

Thm. 5.3 can then be invoked to conclude that every solution $x : \mathcal{I}_x \rightarrow \mathbb{R}$ to (1) satisfies $\dot{x}(t) \in \tilde{F}_{\{U\}}(x(t), t) = G_U^F(x(t), t)$, for almost all $t \in \mathcal{I}_x$. \triangle

6. GENERALIZED TIME DERIVATIVES

Given a countable collection $\mathcal{U} \subset \text{Lip}(\Omega, \mathbb{R})$ of regular functions and a set-valued map $F : \Omega \rightrightarrows \mathbb{R}^n$ with compact values, Prop. 5.1 and Thm. 5.3 suggest the following notion of a generalized derivative of V in the direction(s) F .

Definition 6.1. The \mathcal{U} -generalized derivative of $V \in \text{Lip}(\Omega, \mathbb{R})$ in the direction(s) F , denoted by $\dot{\bar{V}}_{\mathcal{U}} : \Omega \rightarrow \mathbb{R}$ is defined, $\forall (x, t) \in \Omega$, as

$$\dot{\bar{V}}_{\mathcal{U}}(x, t) := \min_{p \in \partial V(x, t)} \max_{q \in \tilde{F}_{\mathcal{U}}(x, t)} p^T [q ; 1], \quad (5)$$

if V is regular, and

$$\dot{\bar{V}}_{\mathcal{U}}(x, t) := \max_{p \in \partial V(x, t)} \max_{q \in \tilde{F}_{\mathcal{U}}(x, t)} p^T [q ; 1], \quad (6)$$

if V is not regular. The \mathcal{U} -generalized derivative is understood to be $-\infty$ when $\tilde{F}_{\mathcal{U}}(x, t)$ is empty. \triangle

Def. 6.1 facilitates a unified treatment of Lyapunov stability theory using regular as well as nonregular candidate Lyapunov functions. A candidate Lyapunov function will be called a Lyapunov function if the \mathcal{U} -generalized derivative is negative.

Definition 6.2. If $V \in \text{Lip}(\Omega, \mathbb{R})$ is positive definite and if $\dot{\bar{V}}_{\mathcal{U}}(x, t) \leq 0, \forall (x, t) \in \Omega$, then V is called a \mathcal{U} -generalized Lyapunov function for (1). \triangle

If V is regular, then it can be assumed, without loss of generality, that $V \in \mathcal{U}$. In this case, $\tilde{F}_{\mathcal{U}} \subseteq G_V^F$, and hence, $\dot{\bar{V}}_{\mathcal{U}}(x, t) \leq \max \dot{\bar{V}}(x, t), \forall (x, t) \in \Omega$. Thus, by judicious selection of the functions in \mathcal{U} , $\dot{\bar{V}}_{\mathcal{U}}(x, t)$ can be constructed to be less conservative than the set-valued derivatives in [6] and [7]. Naturally, if $\mathcal{U} = \{V\}$ then $\dot{\bar{V}}_{\mathcal{U}} = \dot{\bar{V}}$.

In general, the \mathcal{U} -generalized derivative does not satisfy the chain rule as stated in Prop. 4.2. However, it satisfies the following *weak chain rule* which turns out to be sufficient for Lyapunov-based analysis of differential inclusions.

Theorem 6.3. If $V \in \text{Lip}(\Omega, \mathbb{R})$ and $\mathcal{S}(\Omega) \neq \emptyset$, then $\forall x(\cdot) \in \mathcal{S}(\Omega)$,

$$\dot{V}(x(t), t) \in (\partial V(x(t), t))^T \begin{bmatrix} \tilde{F}_{\mathcal{U}}(x(t), t) \\ \{1\} \end{bmatrix}, \quad (7)$$

for almost all $t \in \mathcal{I}_x$. In addition, if there exists a function $W : \Omega \rightarrow \mathbb{R}$ such that $\dot{\bar{V}}_{\mathcal{U}}(x, t) \leq W(x, t), \forall (x, t) \in \Omega$, then $\dot{V}(x(t), t) \leq W(x(t), t)$, for almost all $t \in \mathcal{I}_x$.

Proof. Let $x(\cdot) \in \mathcal{S}(\Omega)$. Consider a set of times $\mathcal{T} \subseteq \mathcal{I}_x$ where $\dot{x}(t)$, $\dot{V}(x(t), t)$, and $\dot{U}_i(x(t), t)$ are defined $\forall i \geq 0$ and $\dot{x}(t) \in \tilde{F}_{\mathcal{U}}(x(t), t)$. Using Thm. 5.3 and the facts that $x(\cdot)$ is absolutely continuous and V is locally Lipschitz, it can be concluded that $\mu(\mathcal{I}_x \setminus \mathcal{T}) = 0$.

If V is regular, then arguments similar to the proof of Thm. 5.3 can be used to conclude that $\dot{V}(x(t), t) = p^T [\dot{x}(t) ; 1], \forall p \in \partial V(x(t), t), \forall t \in \mathcal{T}$. Thus, (5) and Thm. 5.3 imply that $\dot{V}(x(t), t) \in (\partial V(x(t), t))^T \left[\tilde{F}_{\mathcal{U}}(x(t), t) ; \{1\} \right]$ and $\dot{V}(x(t), t) \leq W(x(t), t)$, for almost all $t \in \mathcal{I}_x$.

If V is not regular, then [9, Prop. 4] (see also, [28, Thm. 2]) can be used to conclude that, for almost every $t \in \mathcal{I}_x$, $\exists p_0 \in \partial V(x(t), t)$ such that $\dot{V}(x(t), t) = p_0^T [\dot{x}(t) ; 1]$. Thus, (6) and Thm. 5.3 imply that $\dot{V}(x(t), t) \in (\partial V(x(t), t))^T \left[\tilde{F}_{\mathcal{U}}(x(t), t) ; \{1\} \right]$ and $\dot{V}(x(t), t) \leq W(x(t), t)$ for almost all $t \in \mathcal{I}_x$. \square

The following sections develop relaxed Lyapunov-like stability theorems for differential inclusions based on the properties of the \mathcal{U} -generalized derivative hitherto established.

7. STABILITY OF AUTONOMOUS SYSTEMS

In this section, \mathcal{U} -generalized Lyapunov functions are utilized to formulate less conservative extensions to stability and invariance results for autonomous differential inclusions of the form

$$\dot{x} \in F(x), \quad (8)$$

where $F : \mathcal{D} \rightrightarrows \mathbb{R}^n$ is a set-valued map.

A. Lyapunov stability

The following Lyapunov stability theorem is a consequence of Thm. 6.3.

Theorem 7.1. *Let $0 \in \mathcal{D}$ and let $F : \mathcal{D} \rightrightarrows \mathbb{R}^n$ be a locally bounded set-valued map with compact values such that (8) admits local solutions over \mathcal{D} . If there exists a countable collection, $\mathcal{U} \subset \text{Lip}(\Omega, \mathbb{R})$, of regular functions and a \mathcal{U} -generalized Lyapunov function $V : \mathcal{D} \rightarrow \mathbb{R}$ for (8), then (8) is stable at $x = 0$. In addition, if $\dot{\bar{V}}_{\mathcal{U}}(x) \leq -W(x)$, $\forall x \in \mathcal{D}$, for some positive definite function $W \in \mathcal{C}^0(\mathcal{D}, \mathbb{R})$, then (8) is asymptotically stable at $x = 0$. Furthermore, if $\mathcal{D} = \mathbb{R}^n$ and if the sublevel sets $L_l := \{x \in \mathbb{R}^n \mid V(x) \leq l\}$ are compact for all $l \in \mathbb{R}_{>0}$, then (8) is globally asymptotically stable at $x = 0$.*

Proof. Given $\epsilon > 0$, let $r > 0$ be such that $\bar{\mathbb{B}}(0, r) \subset \mathcal{D}$ and $r \in (0, \epsilon]$. Let $\beta \in [0, \min_{\|x\|=r} V(x))$ and $L_\beta := \{x \in \bar{\mathbb{B}}(0, r) \mid V(x) \leq \beta\}$. Since V is continuous, $\exists \delta > 0$ such that $\bar{\mathbb{B}}(0, \delta) \subset L_\beta$. Let $x(\cdot) \in \mathcal{S}(\mathcal{D})$ be a solution to (8). Using Thm. 6.3, which implies that $t \mapsto V(x(t))$ is nonincreasing on \mathcal{I}_x , and standard arguments (see, e.g., [29, Thm. 4.8]), it can be shown that L_β is compact, (strongly) forward invariant, and $L_\beta \subset \mathcal{D}$. Hence, every solution $x(\cdot) \in \mathcal{S}(L_\beta)$ is precompact, and by Lemma 3.3, complete. Furthermore, if $x(\cdot) \in \mathcal{S}(\bar{\mathbb{B}}(0, \delta))$ then $x(t) \in \bar{\mathbb{B}}(0, \epsilon)$, $\forall t \in \mathbb{R}_{\geq t_0}$.

In addition, if $\dot{\bar{V}}_{\mathcal{U}}(x) \leq -W(x)$, $\forall x \in \mathcal{D}$, for some positive definite function $W \in \mathcal{C}^0(\mathcal{D}, \mathbb{R})$, then Thm. 6.3 implies that $t \mapsto V(x(t))$ is strictly decreasing on $\mathbb{R}_{\geq t_0}$ provided $x(t_0) \in \mathbb{B}(0, \delta)$. Asymptotic stability and global asymptotic stability (in the case where the sublevel sets of V are compact) of (8) at $x = 0$ then follow from standard arguments (see, e.g., [27, Section 5.3.2]). \square

The following example presents a case where tests based on $\dot{\bar{V}}$ and $\dot{\bar{V}}$ are inconclusive but Thm. 7.1 can be used to establish asymptotic stability.

Example 7.2. Let $H : \mathbb{R} \rightrightarrows \mathbb{R}$ be defined as

$$H(y) := \begin{cases} \{0\} & |y| \neq 1, \\ [-1, 1] & |y| = 1. \end{cases}$$

Let $F : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ be defined as

$$F(x) := \begin{bmatrix} \{-x_1 + x_2\} + H(x_2) \\ \{-x_1 - x_2\} + H(x_1) \end{bmatrix}.$$

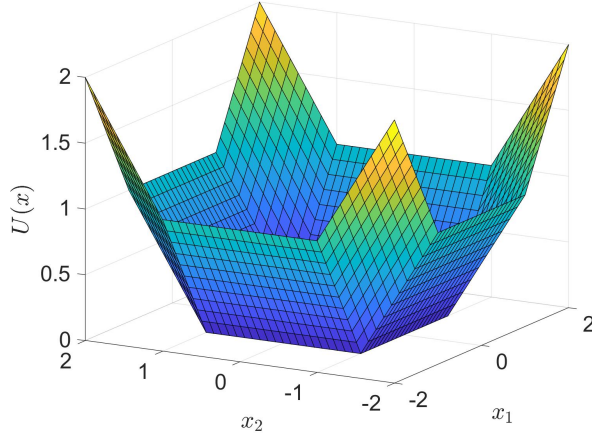


Fig. 1. The function $U : \mathbb{R}^2 \rightarrow \mathbb{R}$.

Consider the differential inclusion in (8) and the candidate Lyapunov function $V : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined as $V(x) := \frac{1}{2} \|x\|_2^2$. Since $V \in C^1(\mathbb{R}^2, \mathbb{R})$, the set-valued derivatives $\dot{\bar{V}}$ in [7] and $\dot{\check{V}}$ in [6] are bounded by

$$\dot{\bar{V}}(x), \dot{\check{V}}(x) \leq \{-x_1^2 - x_2^2\} + x_1 H(x_2) + x_2 H(x_1), \forall x \in \mathbb{R}^2. \quad (9)$$

Since neither $\dot{\check{V}}$ nor $\dot{\bar{V}}$ can be shown to be negative semidefinite everywhere, the inequality in (9) is insufficient to draw conclusions regarding the stability of (8).

The function $U : \mathbb{R}^2 \rightarrow \mathbb{R}$, defined as (see Fig. 1)

$$U(x) := \max((x_1 - 1), 0) - \min((x_1 + 1), 0) + \max((x_2 - 1), 0) - \min((x_2 + 1), 0), \forall x \in \mathbb{R}^2,$$

satisfies $U \in \text{Lip}(\mathbb{R}^2, \mathbb{R})$. In addition, since U is convex, it is also regular [26, Prop. 2.3.6]. The Clarke gradient of U is given by,

$$\partial U(x) = \begin{cases} \{\text{sgn } 1(x_1)\} ; \{\text{sgn } 1(x_2)\} & |x_1| \neq 1 \wedge |x_2| \neq 1, \\ [\overline{\text{co}}\{0, \text{sgn}(x_1)\} ; \{\text{sgn } 1(x_2)\}] & |x_1| = 1 \wedge |x_2| \neq 1, \\ \{\text{sgn } 1(x_1)\} ; \overline{\text{co}}\{0, \text{sgn}(x_2)\}] & |x_1| \neq 1 \wedge |x_2| = 1, \\ [\overline{\text{co}}\{0, \text{sgn}(x_1)\} ; \overline{\text{co}}\{0, \text{sgn}(x_2)\}] & |x_1| = 1 \wedge |x_2| = 1, \end{cases}$$

where

$$\text{sgn } 1(y) := \begin{cases} 0 & -1 < y < 1, \\ \text{sgn}(y) & \text{otherwise.} \end{cases}$$

In this case, the reduced inclusion G_U^F is given by

$$G_U^F(x) = \begin{cases} F(x) & |x_1| \neq 1 \wedge |x_2| \neq 1, \\ \emptyset & \text{otherwise.} \end{cases}$$

Since $G_U^F(x) \subset F(x), \forall x \in \mathbb{R}^2, \tilde{F}_{\{U\}} = G_U^F$. Since $V \in C^1(\mathbb{R}^2, \mathbb{R}), \partial V(x) = \left\{ \frac{\partial V}{\partial x}(x) \right\}$, and hence, the $\{U\}$ -generalized derivative of V in the direction(s) F is given by

$$\begin{aligned} \dot{\bar{V}}_{\{U\}}(x) &= \max_{q \in \tilde{F}_{\{U\}}(x)} \left(\frac{\partial V}{\partial x}(x) \right)^T q, \forall x \in \mathbb{R}^2, \\ &= \begin{cases} [x_1 & x_2] [-x_1 + x_2; -x_1 - x_2] & |x_1| \neq 1 \wedge |x_2| \neq 1, \\ -\infty & \text{otherwise,} \end{cases} \\ &\leq -x_1^2 - x_2^2, \forall x \in \mathbb{R}^2. \end{aligned}$$

Global asymptotic stability of (8) at $x = 0$ then follows from Thm. 7.1. \triangle

In applications where a negative definite bound on the derivative of the candidate Lyapunov function cannot be found easily, the invariance principle is invoked. The following section develops invariance results using \mathcal{U} -generalized derivatives.

B. Invariance principle

Analogous of the Barbashin-Krasovskii-LaSalle invariance principle for autonomous differential inclusions appear in results such as [7], [13], [30]. Estimates of the limiting invariant set that are less conservative than those developed in [7], [13], [30] can be obtained by using locally Lipschitz, regular functions to reduce the admissible directions in F . For example, the following theorem extends the invariance principle developed by Bacciotti and Ceragioli (see [7, Thm. 3]).

Theorem 7.3. *Let $F : \mathcal{D} \rightrightarrows \mathbb{R}^n$ be locally bounded and outer semicontinuous [25, Def. 5.4] over \mathcal{D} ; $F(x)$ be nonempty, convex, and compact, $\forall x \in \mathcal{D}$; $\mathcal{U} \subset \text{Lip}(\mathcal{D}, \mathbb{R})$ be a countable collection of regular functions; $C \subset \mathcal{D}$ be a compact set that is strongly forward invariant with respect to (8); $V \in \text{Lip}(\mathcal{D}, \mathbb{R})$; and $\dot{\bar{V}}_{\mathcal{U}}(x) \leq 0, \forall x \in \mathcal{D}$. If M is the largest weakly forward invariant set (with respect to (8)) in $\bar{E} \cap C$, where $E := \left\{ x \in \mathcal{D} \mid \dot{\bar{V}}_{\mathcal{U}}(x) = 0 \right\}$, then every $x(\cdot) \in \mathcal{S}(C)$ is complete and satisfies $\lim_{t \rightarrow \infty} \text{dist}(x(t), M) = 0$.*

Proof. Existence of at least one $x(\cdot) \in \mathcal{S}(C)$ follows from [1, §7, Thm. 1] and completeness of every $x(\cdot) \in \mathcal{S}(C)$ follows from Lemma 3.3. Given any $x(\cdot) \in \mathcal{S}(C)$, the same argument as [7, Thm. 3] indicates that $t \mapsto V(x(t))$ is constant on $\omega(x(\cdot))$, the ω -limit set of $x(\cdot)$ [7, Def. 3]. Note that $\omega(x(\cdot))$ is weakly invariant [13, Prop. 2.8] (see also, [1, §12, Lem. 4]) and $\omega(x(\cdot)) \subset C$. Let $y : \mathcal{I} \rightarrow \omega(x(\cdot))$ be a solution to (8) such that $y(t_0) \in \omega(x)$. The existence of such a solution follows from weak invariance of $\omega(x(\cdot))$. Since $t \mapsto V(x(t))$ is constant on $\omega(x(\cdot))$, $\dot{V}(y(t)) = 0, \forall t \in \mathbb{R}_{\geq t_0}$.

Let \mathcal{T} be a set of time instances where $\dot{y}(t)$ is defined and $\dot{y}(t) \in \tilde{F}_{\mathcal{U}}(y(t))$. If V is regular, then arguments similar to the proof of Thm. 5.3 can be used to conclude that $\forall t \in \mathcal{T}$ and $\forall p \in \partial V(y(t)), 0 = \dot{V}(y(t)) = p^T \dot{y}(t)$. Since $\dot{\bar{V}}_{\mathcal{U}}(x) = \min_{p \in \partial V(x)} \max_{q \in \tilde{F}_{\mathcal{U}}(x)} p^T q \leq 0, \forall x \in \mathcal{D}$ and $p^T \dot{y}(t) = 0, \forall p \in \partial V(y(t))$, it follows that $\dot{\bar{V}}_{\mathcal{U}}(y(t)) = 0, \forall t \in \mathcal{T}$, which means that $y(t) \in E$, for almost all $t \in \mathbb{R}_{\geq t_0}$.

If V is not regular then [9, Prop. 4] (see also, [28, Thm. 2]) can be used to conclude that for almost every $t \in \mathcal{I}$, $\exists p_0 \in \partial V(y(t))$ such that $0 = \dot{V}(y(t)) = p_0^T \dot{y}(t)$. Since, $\dot{\bar{V}}_{\mathcal{U}}(x) = \max_{p \in \partial V(x)} \max_{q \in \tilde{F}_{\mathcal{U}}(x)} p^T q \leq 0, \forall x \in \mathcal{D}$

and $p_0^T \dot{y}(t) = 0$ for some $p_0 \in \partial V(y(t))$, it follows that $\dot{\bar{V}}_{\mathcal{U}}(y(t)) = 0, \forall t \in \mathcal{T}$, which means that $y(t) \in E$ for almost all $t \in \mathbb{R}_{\geq t_0}$.

Since $y(\cdot) \in \mathcal{C}^0(\mathbb{R}_{\geq t_0}, \mathbb{R}^n)$, $y(t) \in \bar{E}, \forall t \in \mathbb{R}_{\geq t_0}$. That is, $\omega(x(\cdot)) \subset \bar{E}$, and hence, $\omega(x(\cdot)) \subset \bar{E} \cap C$. Since $\omega(x(\cdot))$ is weakly invariant, $\omega(x(\cdot)) \subset M$. As a result, $\lim_{t \rightarrow \infty} \text{dist}(x(t), \omega(x(\cdot))) = 0$ implies $\lim_{t \rightarrow \infty} \text{dist}(x(t), M) = 0$. \square

The following corollary illustrates one of the many alternative ways to establish the existence of a compact strongly forward invariant set needed to apply Thm. 7.3.

Corollary 7.4. *Let $F : \mathcal{D} \rightrightarrows \mathbb{R}^n$ be locally bounded and outer semicontinuous [25, Def. 5.4] over \mathcal{D} ; $F(x)$ be nonempty, convex, and compact, $\forall x \in \mathcal{D}$; and $\mathcal{U} \subset \text{Lip}(\mathcal{D}, \mathbb{R})$ be a countable collection of regular functions. Let $V \in \text{Lip}(\mathcal{D}, \mathbb{R})$ and $\dot{\bar{V}}_{\mathcal{U}}(x) \leq 0, \forall x \in \mathcal{D}$. Let $l > 0$ be such that the level set $L_l := \{x \in \mathcal{D} \mid V(x) \leq l\}$ is closed and a connected component C_l of L_l is bounded. If M is the largest weakly forward invariant set contained in $\bar{E} \cap \bar{C}_l$, where $E := \{x \in \mathcal{D} \mid \dot{\bar{V}}_{\mathcal{U}}(x) = 0\}$, then every $x(\cdot) \in \mathcal{S}(\bar{C}_l)$ is complete and satisfies $\lim_{t \rightarrow \infty} \text{dist}(x(t), M) = 0$.*

Proof. Existence of at least one $x(\cdot) \in \mathcal{S}(\bar{C}_l)$ follows from [1, §7, Thm. 1]. Given $x(\cdot) \in \mathcal{S}(\bar{C}_l)$, it can be concluded from Thm. 6.3 that $t \mapsto V(x(t))$ is nonincreasing on \mathcal{I}_x . Note that by definition of \mathcal{I}_x , $x(t) \in \mathcal{D}$, for all $t \in \mathcal{I}_x$. If there exists $t_1 \in \mathcal{I}_x$ such that $x(t_1) \notin \bar{C}_l$, then continuity of $x(\cdot)$, the fact that C_l is a connected component of L_l , and the fact that L_l is closed imply that $x(t_1) \notin L_l$, which is impossible since $t \mapsto V(x(t))$ is nonincreasing on \mathcal{I}_x . Hence, \bar{C}_l is strongly forward invariant on \mathcal{I}_x . Since C_l is bounded by assumption, \bar{C}_l is also bounded, and as a result, $x(\cdot)$ is precompact. Therefore, by Lemma 3.3, $x(\cdot)$ is complete. The conclusion of the corollary then follows from Thm. 7.3 with $C = \bar{C}_l$. \square

The invariance principle is often applied to conclude asymptotic stability at the origin in the form of the following corollary.

Corollary 7.5. *Let $0 \in \mathcal{D}$ and $V \in \text{Lip}(\mathcal{D}, \mathbb{R})$ be a positive definite function. Let $F : \mathcal{D} \rightrightarrows \mathbb{R}^n$ be locally bounded and outer semicontinuous [25, Def. 5.4] over \mathcal{D} ; $F(x)$ be nonempty, convex, and compact, $\forall x \in \mathcal{D}$; and $\mathcal{U} \subset \text{Lip}(\mathcal{D}, \mathbb{R})$ be a countable collection of regular functions. If $\dot{\bar{V}}_{\mathcal{U}}(x) \leq 0, \forall x \in \mathcal{D}$, and if, for each $\nu > 0$, no complete solution to (8) remains in the level set $\{x \in \mathcal{D} \mid V(x) = \nu\}$, then (8) is asymptotically stable at $x = 0$. In addition, if $\mathcal{D} = \mathbb{R}^n$ and the sublevel sets $\{x \in \mathbb{R}^n \mid V(x) \leq l\}$ are compact for all $l \in \mathbb{R}_{\geq 0}$, then (8) is globally asymptotically stable at $x = 0$.*

Proof. To prove the corollary, it is first established that every complete solution converges to the origin, and then it is shown that all solutions that start in a small neighborhood of the origin are complete, and hence, converge to the origin.

Given a complete solution $x(\cdot)$, since $t \mapsto V(x(t))$ is decreasing (by Thm. 6.3) and bounded below (because V is positive definite), $\lim_{t \rightarrow \infty} V(x(t)) = c$ for some $c \geq 0$. For all $y^* \in \omega(x(\cdot))$, $\exists \{t_i\}_{i \in \mathbb{N}} \subset \mathbb{R}_{\geq t_0}$, $\lim_{i \rightarrow \infty} t_i = \infty$

such that $\lim_{i \rightarrow \infty} x(t_i) = y^*$. Since V is continuous, $V(\omega(x(\cdot))) = \{c\}$. If it can be shown that $c = 0$, then positive definiteness of V would imply that $\omega(x(\cdot)) = \{0\}$, and hence, $\lim_{t \rightarrow \infty} x(t) = 0$.

To prove that $c = 0$ using contradiction, assume that there exists a complete solution $x : \mathbb{R}_{\geq t_0} \rightarrow \mathbb{R}^n$ such that $c > 0$. Let $y : \mathbb{R}_{\geq t_0} \rightarrow \mathbb{R}^n$ be a solution to (8) such that $y(t) \in \omega(x(\cdot))$, $\forall t \in \mathbb{R}_{\geq t_0}$. Such a solution exists since $\omega(x(\cdot))$ is weakly invariant. Along the solution y , $V(y(t)) = c > 0$, $\forall t \in \mathbb{R}_{\geq t_0}$, which contradicts the hypothesis that no complete solution to (8) remains in the level set $\{x \in \mathcal{D} \mid V(x) = \nu\}$ for any $\nu > 0$. Therefore, $c = 0$, and hence, every complete solution converges to the origin.

Select $r > 0$ such that $\bar{B}(0, r) \subset \mathcal{D}$ and $\delta > 0$ such that $\bar{B}(0, \delta) \subseteq L_\beta := \{x \in \bar{B}(0, r) \mid V(x) \leq \beta\}$, where $\beta \in [0, \min_{\|x\|=r} V(x))$. Lyapunov stability of (8) at $x = 0$ and the fact that L_β is strongly forward invariant on \mathcal{I}_x follows from Thm. 7.1. Since L_β is compact, all solutions starting in L_β are precompact, and hence, complete, by Lemma 3.3. Since all complete solutions converge to the origin, $x(\cdot) \in \mathcal{S}(\bar{B}(0, \delta)) \implies \lim_{t \rightarrow \infty} x(t) = 0$; hence, (8) is asymptotically stable at $x = 0$. If $\mathcal{D} = \mathbb{R}^n$ and the sublevel sets $\{x \in \mathbb{R}^n \mid V(x) \leq l\}$ are compact for all $l \in \mathbb{R}_{\geq 0}$, then r , and hence, δ , can be selected arbitrarily large; therefore, (8) is globally asymptotically stable at $x = 0$. \square

The following example demonstrates the utility of the developed invariance principle.

Example 7.6. Let $H : \mathbb{R} \rightrightarrows \mathbb{R}$ be defined as

$$H(y) := \begin{cases} \{0\} & |y| \neq 1, \\ [-\frac{1}{2}, \frac{1}{2}] & |y| = 1, \end{cases}$$

and let $F : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ be defined as

$$F(x) := \begin{bmatrix} \{x_2\} + H(x_2) \\ \{-x_1 - x_2\} + H(x_1) \end{bmatrix}, \forall x \in \mathbb{R}^2, \quad (10)$$

and consider the differential inclusion in (8). The candidate Lyapunov function $V : \mathbb{R}^2 \rightarrow \mathbb{R}$ is defined as $V(x) := 1/2 \|x\|_2^2$, $\forall x \in \mathbb{R}^2$. Since $V \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R})$, the set-valued derivatives $\dot{\bar{V}}$ in [7] and $\dot{\check{V}}$ in [6] are bounded by

$$\dot{\bar{V}}(x), \dot{\check{V}}(x) \leq \{-x_2^2\} + x_2 H(x_1) + x_1 H(x_2), \forall x \in \mathbb{R}^2. \quad (11)$$

That is, neither $\dot{\bar{V}}(x)$ nor $\dot{\check{V}}(x)$ are negative semidefinite everywhere, and hence, the inequality in (11) is inconclusive.

Let $U : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined as in Example 7.2. The $\{U\}$ -reduced set-valued map corresponding to F is given by

$$\tilde{F}_{\{U\}}(x) = \begin{cases} F(x) & |x_1| \neq 1 \wedge |x_2| \neq 1, \\ [\{0\}; (\{-1\} + [-\frac{1}{2}, \frac{1}{2}])] & x_1 = 1 \wedge x_2 = 0, \\ [\{0\}; (\{1\} + [-\frac{1}{2}, \frac{1}{2}])] & x_1 = -1 \wedge x_2 = 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

The $\{U\}$ –generalized derivative of V in the direction(s) F is then given by

$$\begin{aligned} \dot{\bar{V}}_{\{U\}}(x) &:= \max_{q \in \bar{F}_{\{U\}}(x)} [x_1 \ x_2] q, \forall x \in \mathbb{R}^2 \\ &= \begin{cases} [x_1 \ x_2] [x_2 \ ; \ -x_1 - x_2] & |x_1| \neq 1 \wedge |x_2| \neq 1, \\ \max [1 \ 0] [\{0\} \ ; \ [-\frac{3}{2}, -\frac{1}{2}]] & x_1 = 1 \wedge x_2 = 0, \\ \max [-1 \ 0] [\{0\} \ ; \ [\frac{1}{2}, \frac{3}{2}]] & x_1 = -1 \wedge x_2 = 0, \\ -\infty & \text{otherwise,} \end{cases} \\ &\leq -x_2^2, \forall x \in \mathbb{R}^2. \end{aligned}$$

In this case, the set E in Corollary 7.4 is given by $E = \{x \in \mathbb{R}^2 \mid x_2 = 0\}$. Since the level sets L_l are bounded and connected, $\forall l \in \mathbb{R}_{\geq 0}$, and since the largest invariant set contained within $\bar{E} \cap L_l$ is $\{[0 \ ; \ 0]\}$, $\forall l \in \mathbb{R}_{\geq 0}$, Corollary 7.4 can be invoked to conclude that all solutions to (8) converge to the origin.

From Thm. 6.3, $\dot{V}(x(t), t) \leq -x_2^2(t)$ for almost all $t \in \mathbb{R}_{\geq 0}$; hence, given any $\nu > 0$, a trajectory of (8) can remain on the level set $\{x \in \mathbb{R}^n \mid V(x) = \nu\}$ if and only if $x_2(t) = 0$ and $x_1(t) = \pm\sqrt{2\nu}$, for all $t \in \mathbb{R}_{\geq 0}$. From Thm. 5.3, the state $[x_1 \ ; \ x_2]$ can remain constant at $[\pm\sqrt{2\nu} \ ; \ 0]$ for all $t \in \mathbb{R}_{\geq 0}$ only if $[0 \ ; \ 0] \in F([\pm\sqrt{2\nu} \ ; \ 0])$, which is not true for the inclusion in (10). Therefore, Corollary 7.5 can be invoked to conclude that the system is globally asymptotically stable at $x = 0$. \triangle

8. STABILITY OF NONAUTONOMOUS SYSTEMS

In this section, \mathcal{U} –generalized derivatives are used to establish the following forms of uniform and asymptotic stability.

Definition 8.1. The differential inclusion in (1) is said to be (strongly)

- (a) *uniformly stable* at $x = 0$, if $\forall \epsilon > 0 \exists \delta > 0$ such that every $x(\cdot) \in \mathcal{S}(\bar{\mathbb{B}}(0, \delta) \times \mathbb{R}_{\geq 0})$ is complete and satisfies $x(t) \in \bar{\mathbb{B}}(0, \epsilon)$, $\forall t \in \mathbb{R}_{\geq t_0}$.
- (b) *globally uniformly stable* at $x = 0$, if it is uniformly stable at $x = 0$ and $\forall \epsilon > 0 \exists \Delta > 0$ such that every $x(\cdot) \in \mathcal{S}(\bar{\mathbb{B}}(0, \epsilon) \times \mathbb{R}_{\geq 0})$ is complete and satisfies $x(t) \in \bar{\mathbb{B}}(0, \Delta)$, $\forall t \in \mathbb{R}_{\geq t_0}$.
- (c) *uniformly asymptotically stable* at $x = 0$ if it is uniformly stable at $x = 0$ and $\exists c > 0$ such that $\forall \epsilon > 0 \exists T \geq 0$ such that every $x(\cdot) \in \mathcal{S}(\bar{\mathbb{B}}(0, c) \times \mathbb{R}_{\geq 0})$ is complete and satisfies $x(t) \in \bar{\mathbb{B}}(0, \epsilon)$, $\forall t \in \mathbb{R}_{\geq t_0+T}$.
- (d) *globally uniformly asymptotically stable* at $x = 0$ if it is uniformly stable at $x = 0$ and $\forall c, \epsilon > 0 \exists T \geq 0$ such that every $x(\cdot) \in \mathcal{S}(\bar{\mathbb{B}}(0, c) \times \mathbb{R}_{\geq 0})$ is complete and satisfies $x(t) \in \bar{\mathbb{B}}(0, \epsilon)$, $\forall t \in \mathbb{R}_{\geq t_0+T}$. \triangle

While the results in this section are stated in terms of stability of the state at the origin and uniformity with respect to time, they extend in a straightforward manner to partial stability and uniformity with respect to a part of the state (see, e.g., [16, Def. 4.1]), and stability of arbitrary compact sets.

A. Lyapunov stability

The following fundamental Lyapunov-based stability result demonstrates the utility of \mathcal{U} –generalized derivatives.

Theorem 8.2. Let $0 \in \mathcal{D}$ and let $F : \Omega \rightrightarrows \mathbb{R}^n$ be a locally bounded set-valued map with compact values such that (1) admits local solutions over Ω . If there exists a positive definite function $V \in \text{Lip}(\Omega, \mathbb{R})$, a pair of positive definite functions $\overline{W}, \underline{W} \in \mathcal{C}^0(\mathcal{D}, \mathbb{R})$, and a countable collection $\mathcal{U} \subset \text{Lip}(\Omega, \mathbb{R})$ of regular functions, such that

$$\begin{aligned} \underline{W}(x) &\leq V(x, t) \leq \overline{W}(x), \quad \forall (x, t) \in \Omega, \\ \dot{\overline{V}}_{\mathcal{U}}(x, t) &\leq 0, \quad \text{for all } x \in \mathcal{D}, \text{ and almost all } t \in \mathbb{R}_{\geq 0}, \end{aligned} \quad (12)$$

then (1) is uniformly stable at $x = 0$. In addition, if there exists a positive definite function $W \in \mathcal{C}^0(\mathcal{D}, \mathbb{R})$ such that

$$\dot{\overline{V}}_{\mathcal{U}}(x, t) \leq -W(x), \quad (13)$$

for all $x \in \mathcal{D}$ and almost all $t \in \mathbb{R}_{\geq 0}$, then (1) is uniformly asymptotically stable at $x = 0$. Furthermore, if $\mathcal{D} = \mathbb{R}^n$ and if the sublevel sets $\{x \in \mathbb{R}^n \mid \underline{W}(x) \leq c\}$ are compact $\forall c \in \mathbb{R}_{\geq 0}$, then (1) is globally uniformly asymptotically stable at $x = 0$.

Proof. Select $r > 0$ such that $\overline{B}(0, r) \subset \mathcal{D}$. Let $x(\cdot) \in \mathcal{S}(\Omega_c \times \mathbb{R}_{\geq 0})$ where $\Omega_c := \{x \in \overline{B}(0, r) \mid \overline{W}(x) \leq c\}$ for some $c \in [0, \min_{\|x\|_2=r} \underline{W}(x)]$. Using Thm. 6.3 and [17, Lemma 2],

$$V(x(t_0), t_0) \geq V(x(t), t), \quad \forall t \in \mathcal{I}_x. \quad (14)$$

Using (14) and arguments similar to [29, Thm. 4.8], it can be shown that every $x(\cdot) \in \mathcal{S}(\Omega_c \times \mathbb{R}_{\geq 0})$ satisfies $x(t) \in \overline{B}(0, r)$, for all $t \in \mathcal{I}_x$. Therefore, all solutions $x(\cdot) \in \mathcal{S}(\Omega_c \times \mathbb{R}_{\geq 0})$ are precompact, and as a consequence of Lemma 3.3, complete. Since \overline{W} is continuous and positive definite, $\exists \delta > 0$ such that $\overline{B}(0, \delta) \subset \Omega_c$. Since δ is independent of t_0 , uniform stability of (1) at $x = 0$ is established. The rest of the proof is identical to [27, Section 5.3.2], and is therefore omitted. \square

In the following example, tests based on $\dot{\overline{V}}$ and $\dot{\underline{V}}$ are inconclusive, but Thm. 8.2 can be invoked to conclude global uniform asymptotic stability of the origin.

Example 8.3. Let $H : \mathbb{R} \rightrightarrows \mathbb{R}$ be defined as

$$H(y) := \begin{cases} \{0\} & |y| \neq 1, \\ [-\frac{1}{2}, \frac{1}{2}] & |y| = 1, \end{cases}$$

and let $F : \mathbb{R}^2 \times \mathbb{R}_{\geq 0} \rightrightarrows \mathbb{R}^2$ be defined as

$$F(x, t) := \begin{bmatrix} \{-x_1 + x_2(1 + g(t))\} + H(x_2) \\ \{-x_1 - x_2\} + H(x_1) \end{bmatrix}, \quad \forall (x, t) \in \mathbb{R}^2 \times \mathbb{R}_{\geq 0},$$

where $g \in \mathcal{C}^1(\mathbb{R}_{\geq 0}, \mathbb{R})$, $0 \leq g(t) \leq 1, \forall t \in \mathbb{R}_{\geq 0}$ and $\dot{g}(t) \leq g(t), \forall t \in \mathbb{R}_{\geq 0}$. Consider the differential inclusion in (1) and the candidate Lyapunov function $V : \mathbb{R}^2 \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ defined as $V(x, t) := x_1^2 + (1 + g(t))x_2^2$. the candidate Lyapunov function satisfies $\|x\|_2^2 \leq V(x, t) \leq 2\|x\|_2^2, \forall (x, t) \in \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$. In this case, since $V \in \mathcal{C}^1(\mathbb{R}^2 \times \mathbb{R}_{\geq 0}, \mathbb{R})$, similar to [29, Example 4.20], the set-valued derivatives $\dot{\overline{V}}$ in [7] and $\dot{\underline{V}}$ in [6] satisfy the bound $\dot{\overline{V}}(x, t), \dot{\underline{V}}(x, t) \leq \{-2x_1^2 - 2x_2^2\} + 2x_1H(x_2) + 2x_2h(t)H(x_1)$, where $h(t) := 1 + g(t)$ and the

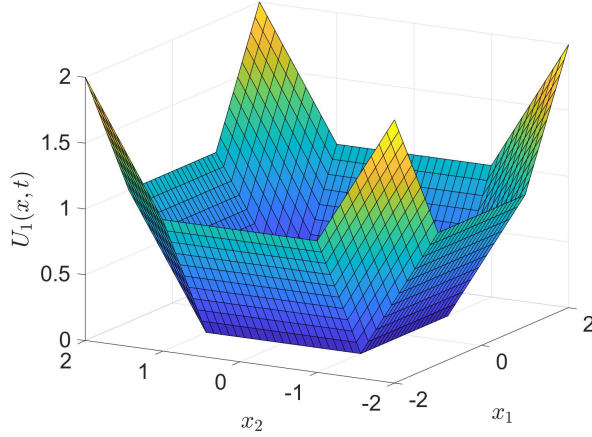


Fig. 2. A snapshot of the function $U_1 : \mathbb{R}^2 \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$.

inequality $2 + 2g(t) - \dot{g}(t) \geq 2$ is utilized. Therefore, neither $\dot{V}(x, t)$ nor $\ddot{V}(x, t)$ can be shown to be negative semidefinite everywhere.

The function $U_1 : \mathbb{R}^2 \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$, defined as (see Fig. 2)

$$U_1(x, t) = \max((x_1 - 1), 0) - \min((x_1 + 1), 0) + \max((x_2 - 1), 0) - \min((x_2 + 1), 0), \forall (x, t) \in \mathbb{R}^2 \times \mathbb{R}_{\geq 0}, \quad (15)$$

satisfies $U_1 \in \text{Lip}(\mathbb{R}^2 \times \mathbb{R}_{\geq 0}, \mathbb{R})$. In addition, since U_1 is convex, it is also regular [26, Prop. 2.3.6]. With

$$\text{sgn } 1(y) := \begin{cases} 0 & -1 < y < 1, \\ \text{sgn}(y) & \text{otherwise,} \end{cases}$$

the Clarke gradient of U_1 is given by

$$\partial U_1(x, t) = \begin{cases} [\{\text{sgn } 1(x_1)\} ; \{\text{sgn } 1(x_2)\} ; \{0\}] & |x_1| \neq 1 \wedge |x_2| \neq 1, \\ [\overline{\text{co}}\{0, \text{sgn}(x_1)\} ; \{\text{sgn } 1(x_2)\} ; \{0\}] & |x_1| = 1 \wedge |x_2| \neq 1, \\ [\{\text{sgn } 1(x_1)\} ; \overline{\text{co}}\{0, \text{sgn}(x_2)\} ; \{0\}] & |x_1| \neq 1 \wedge |x_2| = 1, \\ [\overline{\text{co}}\{0, \text{sgn}(x_1)\} ; \overline{\text{co}}\{0, \text{sgn}(x_2)\} ; \{0\}] & |x_1| = 1 \wedge |x_2| = 1, \end{cases}$$

The $\{U_1\}$ -reduced set-valued map corresponding to F is given by

$$\tilde{F}_{\{U_1\}}(x, t) = \begin{cases} F(x, t) & |x_1| \neq 1 \wedge |x_2| \neq 1, \\ \emptyset & \text{otherwise.} \end{cases}$$

The $\{U_1\}$ –generalized derivative of V in the direction(s) F is then given by

$$\begin{aligned} \dot{\bar{V}}_{\{U_1\}}(x, t) &:= \max_{q \in \bar{F}_{\{U_1\}}(x, t)} \left(\frac{\partial V}{\partial(x, t)}(x, t) \right)^T [q; 1], \forall (x, t) \in \mathbb{R}^2 \times \mathbb{R}_{\geq 0}, \\ &= \begin{cases} [2x_1 & 2x_2 h(t) & \dot{g}(t) x_2^2] [-x_1 + x_2 h(t); -x_1 - x_2; 1] & |x_1| \neq 1 \wedge |x_2| \neq 1, \\ -\infty & \text{otherwise,} \end{cases} \\ &\leq -2 \|x\|_2^2, \forall (x, t) \in \mathbb{R}^2 \times \mathbb{R}_{\geq 0}. \end{aligned}$$

Thm. 8.2 can then be invoked to conclude that (1) is globally uniformly asymptotically stable at $x = 0$. \triangle

B. Invariance-like results

In applications such as adaptive control, Lyapunov methods commonly result in semidefinite Lyapunov functions (i.e., candidate Lyapunov functions with time derivatives bounded by a negative semidefinite function of the state). The following theorem establishes the fact that if the function W in (13) is positive semidefinite then $t \mapsto W(x(t))$ asymptotically decays to zero.

Theorem 8.4. *Let $0 \in \mathcal{D}$, select $r > 0$ such that $\bar{B}(0, r) \subset \mathcal{D}$, and let $F : \Omega \rightrightarrows \mathbb{R}^n$ be a set-valued map with compact values that is locally bounded, uniformly in t , over Ω ,⁵ such that (1) admits local solutions over Ω . If there exists a positive definite function $V \in \text{Lip}(\Omega, \mathbb{R})$, a positive semidefinite function $W \in \mathcal{C}^0(\mathcal{D}, \mathbb{R})$, a pair of positive definite functions $\bar{W}, \underline{W} \in \mathcal{C}^0(\mathcal{D}, \mathbb{R})$, and a countable collection $\mathcal{U} \subset \text{Lip}(\Omega, \mathbb{R})$ of regular functions such that (12) and (13) hold, then every solution $x(\cdot) \in \mathcal{S}(\Omega_c \times \mathbb{R}_{\geq 0})$, with $\Omega_c := \{x \in \bar{B}(0, r) \mid \bar{W}(x) \leq c\}$ and $c \in [0, \min_{\|x\|_2=r} \underline{W}(x))$, is complete, bounded, and satisfies $\lim_{t \rightarrow \infty} W(x(t)) = 0$.*

Proof. Similar to the proof of [17, Corollary 1], it is established that the bounds on $\dot{\bar{V}}_F$ in (5) and (6) imply that V is nonincreasing along all the solutions to (1). The nonincreasing property of V is used to establish boundedness of $x(\cdot)$, which is used to prove the existence and uniform continuity of complete solutions. Barbălat’s lemma [29, Lemma 8.2] is then used to conclude the proof.

Let $x(\cdot) \in \mathcal{S}(\Omega_c \times \mathbb{R}_{\geq 0})$. Using Thm. 6.3 and [17, Lemma 2], $V(x(t_0), t_0) \geq V(x(t), t)$, $\forall t \in \mathcal{I}_x$. Arguments similar to [29, Thm. 4.8] can then be used to show that every $x(\cdot) \in \mathcal{S}(\Omega_c \times \mathbb{R}_{\geq 0})$ satisfies $x(t) \in \bar{B}(0, r)$, $\forall t \in \mathcal{I}_x$. Therefore, all solutions $x(\cdot) \in \mathcal{S}(\Omega_c \times \mathbb{R}_{\geq 0})$ are precompact, and as a consequence of Lemma 3.3, complete.

To establish uniform continuity of the solutions, it is observed that since F is locally bounded, uniformly in t , over Ω , and $x(t) \in \bar{B}(0, r)$ on $\mathbb{R}_{\geq t_0}$, the map $t \mapsto F(x(t), t)$ is uniformly bounded on $\mathbb{R}_{\geq t_0}$. Hence, $\dot{x} \in \mathcal{L}_\infty(\mathbb{R}_{\geq t_0}, \mathbb{R}^n)$. Since $x(\cdot)$ is locally absolutely continuous, $\forall t_1, t_2 \in \mathbb{R}_{\geq t_0}$, $\|x(t_2) - x(t_1)\|_2 = \left\| \int_{t_1}^{t_2} \dot{x}(\tau) d\tau \right\|_2$. Since $\dot{x} \in \mathcal{L}_\infty(\mathbb{R}_{\geq t_0}, \mathbb{R}^n)$, $\left\| \int_{t_1}^{t_2} \dot{x}(\tau) d\tau \right\|_2 \leq \int_{t_1}^{t_2} M d\tau$, where M is a positive constant. Thus, $\|x(t_2) - x(t_1)\|_2 \leq M|t_2 - t_1|$, and hence, $x(\cdot)$ is uniformly continuous on $\mathbb{R}_{\geq t_0}$.

⁵A set-valued map $F : \mathbb{R}^n \times \mathbb{R} \rightrightarrows \mathbb{R}^n$ is locally bounded, uniformly in t , over $\mathcal{D} \times \mathcal{J}$ for some $\mathcal{D} \subseteq \mathbb{R}^n$ and $\mathcal{J} \subseteq \mathbb{R}$, if for every compact $K \subset \mathcal{D}$, there exists $M > 0$ such that $\forall (x, t, y)$ such that $(x, t) \in K \times \mathcal{J}$, and $y \in F(x, t)$, $\|y\|_2 \leq M$.

Since $x \mapsto W(x)$ is continuous and $\bar{B}(0, r)$ is compact, $x \mapsto W(x)$ is uniformly continuous on $\bar{B}(0, r)$. Hence, $t \mapsto W(x(t))$ is uniformly continuous on $\mathbb{R}_{\geq t_0}$. Furthermore, $t \mapsto \int_{t_0}^t W(x(\tau)) d\tau$ is monotonically increasing and from (13), $\int_{t_0}^t W(x(\tau)) d\tau \leq V(x(t_0), t_0) - V(x(t), t) \leq V(x(t_0), t_0)$. Hence, $\lim_{t \rightarrow \infty} \int_{t_0}^t W(x(\tau)) d\tau$ exists and is finite. By Barbălat's Lemma [29, Lemma 8.2], $\lim_{t \rightarrow \infty} W(x(t)) = 0$. \square

In the following example $\dot{\bar{V}}$ and $\dot{\check{V}}$ do not have a negative semidefinite upper bound, but Thm. 8.4 can be invoked to conclude partial stability.

Example 8.5. Let $H : \mathbb{R} \rightrightarrows \mathbb{R}$ be defined as in Example 8.3 and let $F : \mathbb{R}^2 \times \mathbb{R}_{\geq 0} \rightrightarrows \mathbb{R}^2$ be defined as

$$F(x, t) = \begin{bmatrix} \{x_2(1 + g(t))\} + H(x_2) \\ \{-x_1 - x_2\} + H(x_1) \end{bmatrix},$$

where $g \in \mathcal{C}^1(\mathbb{R}_{\geq 0}, \mathbb{R})$, $0 \leq g(t) \leq 1, \forall t \in \mathbb{R}_{\geq 0}$ and $\dot{g}(t) \leq g(t), \forall t \in \mathbb{R}_{\geq 0}$. Consider the differential inclusion in (1) and the candidate Lyapunov function $V : \mathbb{R}^2 \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ defined as $V(x, t) := x_1^2 + (1 + g(t))x_2^2$. The candidate Lyapunov function satisfies $\|x\|_2^2 \leq V(x, t) \leq 2\|x\|_2^2, \forall (x, t) \in \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$. In this case, since $V \in \mathcal{C}^1(\mathbb{R}^2 \times \mathbb{R}_{\geq 0}, \mathbb{R})$, the set-valued derivatives $\dot{\bar{V}}$ in [7] and $\dot{\check{V}}$ in [6] are bounded by

$$\dot{\bar{V}}(x, t), \dot{\check{V}}(x, t) \leq \{-2x_2^2\} + 2x_2h(t)H(x_1) + 2x_1H(x_2), \forall (x, t) \in \mathbb{R}^2 \times \mathbb{R}_{\geq 0},$$

where $h(t) := 1 + g(t)$ and the inequality $2 + 2g(t) - \dot{g}(t) \geq 2$ is utilized. Thus, neither $\dot{\check{V}}$ nor $\dot{\bar{V}}$ are negative semidefinite everywhere.

Let U_1 be defined as in (15). The $\{U_1\}$ -reduced set-valued map corresponding to F is given by

$$\tilde{F}_{\{U_1\}}(x, t) = \begin{cases} F(x, t) & |x_1| \neq 1 \wedge |x_2| \neq 1, \\ [\{0\} ; (\{-1\} + [-\frac{1}{2}, \frac{1}{2}])] & x_1 = 1 \wedge x_2 = 0, \\ [\{0\} ; (\{1\} + [-\frac{1}{2}, \frac{1}{2}])] & x_1 = -1 \wedge x_2 = 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

The $\{U_1\}$ -generalized derivative of V in the direction(s) F is then given by

$$\begin{aligned} \dot{\bar{V}}_{\{U_1\}}(x, t) &:= \max_{q \in \tilde{F}_{\{U_1\}}(x, t)} \left(\frac{\partial V}{\partial(x, t)}(x, t) \right)^T [q ; 1], \forall (x, t) \in \mathbb{R}^2 \times \mathbb{R}_{\geq 0}, \\ &= \begin{cases} [2x_1 \quad 2x_2h(t) \quad \dot{g}(t)x_2^2] [x_2h(t) ; -x_1 - x_2 ; 1] & |x_1| \neq 1 \wedge |x_2| \neq 1, \\ \max [2 \quad 0 \quad 0] [\{0\} ; [\frac{3}{2}, -\frac{1}{2}] ; 1] & x_1 = 1 \wedge x_2 = 0, \\ \max [-2 \quad 0 \quad 0] [\{0\} ; [-\frac{1}{2}, \frac{3}{2}] ; 1] & x_1 = -1 \wedge x_2 = 0, \\ -\infty & \text{otherwise,} \end{cases} \\ &\leq -2x_2^2, \forall (x, t) \in \mathbb{R}^2 \times \mathbb{R}_{\geq 0}. \end{aligned}$$

Thm. 8.4 can then be invoked to conclude that $t \mapsto x_1(t) \in \mathcal{L}_\infty(\mathbb{R}_{\geq t_0}, \mathbb{R})$ and $\lim_{t \rightarrow \infty} x_2(t) = 0$. \triangle

Thm. 8.4 and its counterparts are widely used in applications such as adaptive control to establish stability (but not asymptotic stability) of the state and convergence of a part of the state (e.g., tracking errors, but not parameter

estimation errors) to the origin. Under certain excitation conditions, asymptotic stability (and as a result, convergence of the entire state to the origin) can be established using Matrosov theorems [18].

C. Matrosov theorems

In this section, a less conservative generalization of Matrosov results for uniform asymptotic stability of nonautonomous systems is developed. In particular, the nonsmooth version [22, Thm. 1] of the nested Matrosov theorem [19, Thm. 1] is generalized. The following definitions of Matrosov functions are inspired by [22].

Definition 8.6. Let $\gamma, \delta, \Delta > 0$ be constants. A finite set of functions $\{Y_j\}_{j=1}^M \subset \mathcal{C}^0(\overline{B}(0_m, \gamma) \times D(\delta, \Delta), \mathbb{R})$ is said to have the *Matrosov property* relative to (γ, δ, Δ) if $\forall j \in \{0, \dots, M\}$,

$$((z, x) \in \overline{B}(0_m, \gamma) \times D(\delta, \Delta)) \wedge (Y_i(z, x) = 0, \forall i \in \{0, \dots, j\}) \implies Y_{j+1}(z, x) \leq 0,$$

where $Y_0(z, x) = 0$ and $Y_{M+1}(z, x) = 1, \forall (z, x) \in \overline{B}(0_m, \gamma) \times D(\delta, \Delta)$. △

Definition 8.7. Let $\delta, \Delta > 0$ be constants such that $D(\delta, \Delta) \subset \mathcal{D}$. Let $F : \Omega \rightrightarrows \mathbb{R}^n$ be a set-valued map with compact values. The functions $\{W_j\}_{j=1}^M \subset \text{Lip}(\Omega, \mathbb{R})$ are said to be *\mathcal{U} -reduced Matrosov functions* for (F, δ, Δ) if $\exists \phi : \Omega \rightarrow \mathbb{R}^m, \gamma > 0$, and $\{Y_j\}_{j=1}^M \subset \mathcal{C}^0(\overline{B}(0_m, \gamma) \times D(\delta, \Delta), \mathbb{R})$ such that:

- (a) the set of functions $\{Y_j\}_{j=1}^M$ has the Matrosov property relative to (γ, δ, Δ) ,
- (b) $\forall j \in \{1, \dots, M\}$ and $\forall (x, t) \in D(\delta, \Delta) \times \mathbb{R}_{\geq 0}$, $\max\{|W_j(x, t)|, |\phi(x, t)|\} \leq \gamma$, and
- (c) $\forall j \in \{1, \dots, M\}$ there exists a collection of regular functions $\mathcal{U}_j \subset \text{Lip}(D(\delta, \Delta) \times \mathbb{R}_{\geq 0}, \mathbb{R})$ such that $\forall (x, t) \in D(\delta, \Delta) \times \mathbb{R}_{\geq 0}$, $\overline{W}_{\mathcal{U}_j}(x, t) \leq Y_j(\phi(x, t), x)$. △

The following technical Lemmas aid the proof of the Matrosov theorem.

Lemma 8.8. Given $\delta > 0, \exists \epsilon > 0$ such that

$$((z, x) \in \overline{B}(0_m, \gamma) \times D(\delta, \Delta)) \wedge (Y_j(z, x) = 0, \forall j \in \{1, \dots, M-1\}) \implies Y_M(z, x) \leq -\epsilon.$$

Proof. See [19, Claim 1]. □

Lemma 8.9. Let $l \in \{2, \dots, M\}, \tilde{\epsilon} > 0$, and $\tilde{Y}_l \in \mathcal{C}^0(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{R})$. If

$$((z, x) \in \overline{B}(0_m, \gamma) \times D(\delta, \Delta)) \wedge (Y_j(z, x) = 0, \forall j \in \{1, \dots, l-1\}) \implies \tilde{Y}_l(z, x) \leq -\tilde{\epsilon},$$

then $\exists K_{l-1} > 0$ such that

$$((z, x) \in \overline{B}(0_m, \gamma) \times D(\delta, \Delta)) \wedge (Y_j(z, x) = 0, \forall j \in \{1, \dots, l-2\}) \implies K_{l-1}Y_{l-1}(z, x) + \tilde{Y}_l(z, x) \leq -\frac{\tilde{\epsilon}}{2}.$$

Proof. See [19, Claim 2] □

The Matrosov theorem can now be stated as follows.

Theorem 8.10. Let $0 \in \mathcal{D}$ and let $F : \Omega \rightrightarrows \mathbb{R}^n$ be a set-valued map with compact values such that (1) admits solutions over Ω and is uniformly stable at $x = 0$. If, for each pair of numbers $\delta, \Delta \in \mathbb{R}$, such that $0 \leq \delta \leq \Delta$ and

$D(\delta, \Delta) \subset \mathcal{D}$, there exist \mathcal{U} -reduced Matrosov functions for (F, δ, Δ) , then (1) is uniformly asymptotically stable at $x = 0$. If $\mathcal{D} = \mathbb{R}^n$ and if (1) is uniformly globally stable at $x = 0$ then (1) is uniformly globally asymptotically stable at $x = 0$.

Proof. Select $\Delta > 0$ such that $\bar{B}(0, \Delta) \subset \mathcal{D}$ and let $r > 0$ be such that

$$x(\cdot) \in \mathcal{S}(\bar{B}(0, r) \times \mathbb{R}_{\geq 0}) \implies x(t) \in \bar{B}(0, \Delta), \forall t \in \mathbb{R}_{\geq t_0}. \quad (16)$$

Let $\epsilon \in (0, r)$ and select $\delta > 0$ such that

$$x(\cdot) \in \mathcal{S}(\bar{B}(0, \delta) \times \mathbb{R}_{\geq 0}) \implies x(t) \in \bar{B}(0, \epsilon), \forall t \in \mathbb{R}_{\geq t_0}. \quad (17)$$

By repeated application of Lemmas 8.8 and 8.9 it can be shown that $\forall \delta > 0, \exists \zeta > 0$ and $K_1, \dots, K_{M-1} > 0$ such that $\forall (z, x) \in \bar{B}(0_m, \gamma) \times D(\delta, \Delta)$,

$$Z(z, x) := \sum_{j=1}^{M-1} K_j Y_j(z, x) + Y_M(z, x) \leq -\frac{\zeta}{2^{M-1}}. \quad (18)$$

Let $W \in \text{Lip}(\Omega, \mathbb{R})$ be defined as $W(x, t) := \sum_{j=1}^{M-1} K_j W_j(x, t) + W_M(x, t)$. From Def. 8.7.b,

$$|V(x, t)| \leq \gamma \left(1 + \sum_{j=1}^{M-1} K_j \right) =: \eta. \quad (19)$$

Fix $(x_0, t_0) \in \bar{B}(0, r) \times \mathbb{R}_{\geq 0}$ and $x(\cdot) \in \mathcal{S}(\{(x_0, t_0)\})$. The selection of r in (16) implies that the solution $x(\cdot)$ satisfies $x(t) \in \bar{B}(0, \Delta), \forall t \in \mathbb{R}_{\geq t_0}$. From Def. 8.7.c, $\dot{V}_{\mathcal{U}_j}(x, t) \leq Z(\phi(x, t), x), \forall (x, t) \in D(\delta, \Delta) \times \mathbb{R}_{\geq 0}$, and hence, from Thm. 6.3,

$$\dot{V}(x(t), t) \leq Z(\phi(x(t), t), x(t)), \quad (20)$$

for almost all $t \in x^{-1}(D(\delta, \Delta))$. Using Def. 8.7.b and (18),

$$Z(\phi(x(t), t), x(t)) \leq -\frac{\zeta}{2^{M-1}}, \quad (21)$$

for almost all $t \in x^{-1}(D(\delta, \Delta))$.

Let $T > \frac{2^M \eta}{\zeta}$. The claim is that $\|x(t)\| \leq \epsilon, \forall t \in \mathbb{R}_{\geq t_0 + T}$. If not, then the selection of δ in (17) implies that $x(t) \in D(\delta, \Delta), \forall t \in [t_0, t_0 + T]$. Hence, from (20) and (21),

$$\dot{V}(x(t), t) \leq -\frac{\zeta}{2^{M-1}}, \quad (22)$$

for almost all $t \in [t_0, t_0 + T]$. Integrating (22) and using the bound in (19), $\frac{T\zeta}{2^{M-1}} \leq 2\eta$, which contradicts $T > \frac{2^M \eta}{\zeta}$. Hence, $\forall \epsilon \in (0, r), \exists T > 0$ such that $x(\cdot) \in \mathcal{S}(\bar{B}(0, r) \times \mathbb{R}_{\geq 0}) \implies \|x(t)\| < \epsilon, \forall t \in \mathbb{R}_{\geq t_0 + T}$, i.e., (1) is uniformly asymptotically stable at $x = 0$.

If $\mathcal{D} = \mathbb{R}^n$ and if (1) is uniformly globally stable at $x = 0$ then r can be selected arbitrarily large, and hence, the result is global. \square

The following example demonstrates an application of the Matrosov theorem.

Example 8.11. Let $H : \mathbb{R} \rightrightarrows \mathbb{R}$ be defined as in Example 8.3 and let $F : \mathbb{R}^2 \times \mathbb{R}_{\geq 0} \rightrightarrows \mathbb{R}^2$ be defined as in Example 8.5. Let U_1 be defined as in (15). Let $W_1 : \mathbb{R}^2 \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ be defined as $W_1(x, t) := x_1^2 + (1 + g(t))x_2^2$.

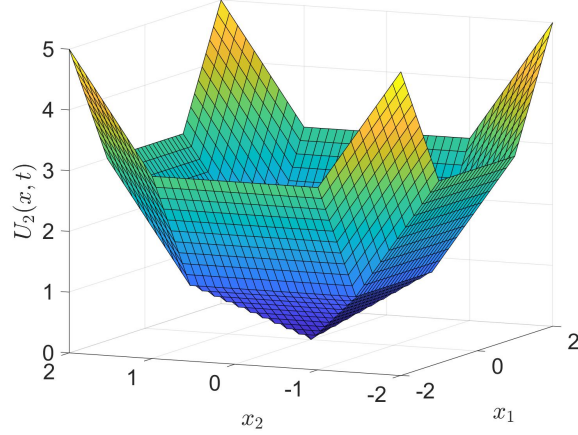


Fig. 3. A snapshot of the function $U_2 : \mathbb{R}^2 \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$.

It follows that $\overline{W}_{1\{U_1\}}(x, t) \leq -2x_2^2$, $\forall (x, t) \in \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$, and uniform global stability of (1) at $x = 0$ can be concluded from Thm. 8.2.

Let $\phi(x, t) = 0$, $\forall (x, t) \in \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$ and let $Y_1(z, x) := -2x_2^2$, $\forall (z, x) \in \mathbb{R} \times \mathbb{R}^2$. Let $W_2(x, t) := x_1x_2$. The function $U_2 : \mathbb{R}^2 \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$, defined as (see Fig. 3)

$$U_2(x, t) = \begin{cases} |x_1| & |x_1| > |x_2| \wedge x \in \text{Sq}, \\ |x_2| & |x_1| \leq |x_2| \wedge x \in \text{Sq}, \\ 1 + U^*(x, t) & x \notin \text{Sq}, \end{cases}$$

where

$$U^*(x, t) = \max((2x_1 - 2), 0) - \min((2x_1 + 2), 0) + \max((2x_2 - 2), 0) - \min((2x_2 + 2), 0), \forall (x, t) \in \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$$

and ‘Sq’ denotes the open unit square centered at the origin, satisfies $U_2 \in \text{Lip}(\mathbb{R}^2 \times \mathbb{R}_{\geq 0}, \mathbb{R})$. In addition, since U_2 is convex, it is also regular [26, Prop. 2.3.6]. The Clarke gradient of U_2 is given by

$$\partial U_2(x, t) = \begin{cases} [\{2 \text{sgn } 1(x_1)\} ; \{2 \text{sgn } 1(x_2)\} ; \{0\}] & |x_1| \neq 1 \wedge |x_2| \neq 1, \\ [\overline{\text{co}}\{0, 2 \text{sgn}(x_1)\} ; \{2 \text{sgn } 1(x_2)\} ; \{0\}] & |x_1| = 1 \wedge |x_2| \neq 1, \\ [\{2 \text{sgn } 1(x_1)\} ; \overline{\text{co}}\{0, 2 \text{sgn}(x_2)\} ; \{0\}] & |x_1| \neq 1 \wedge |x_2| = 1, \end{cases}$$

if $x \notin \overline{\text{Sq}}$,

$$\partial U_2(x, t) = \begin{cases} [[-1, 1] ; [-1, 1] ; \{0\}] & |x_1| = 0 \wedge |x_2| = 0, \\ [\text{sgn}(x_1) ; \{0\} ; \{0\}] & |x_1| > |x_2|, \\ [\{0\} ; \text{sgn}(x_2) ; \{0\}] & |x_1| < |x_2|, \\ [\overline{\text{co}}\{0, \text{sgn}(x_1)\} ; \overline{\text{co}}\{0, \text{sgn}(x_2)\} ; \{0\}] & |x_1| = |x_2| > 0, \end{cases}$$

if $x \in \text{Sq}$, and

$$\partial U_2(x, t) = \begin{cases} [\overline{\text{co}} \{\text{sgn } 1(x_1), 2 \text{sgn } 1(x_1)\} ; \overline{\text{co}} \{\text{sgn } 1(x_2), 2 \text{sgn } 1(x_2)\} ; \{0\}] & |x_1| \neq |x_2|, \\ [\overline{\text{co}} \{\text{sgn}(x_1), 2 \text{sgn}(x_1), 0\} ; \overline{\text{co}} \{\text{sgn}(x_2), 2 \text{sgn}(x_2), 0\} ; \{0\}] & |x_1| = |x_2|, \end{cases}$$

if $x \in \text{bd}(\text{Sq})$. The $\{U_2\}$ -reduced set-valued map corresponding to F is given by

$$\tilde{F}_{\{U_2\}}(x, t) = \begin{cases} F(x, t) & x \notin \overline{\text{Sq}} \wedge |x_1| \neq 1 \wedge |x_2| \neq 1, \\ F(x, t) & x \in \text{Sq} \wedge |x_1| \neq |x_2|, \\ \{0\} & x \in \text{Sq} \wedge |x_1| = 0 \wedge |x_2| = 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

The $\{U_2\}$ -generalized derivative of W_2 is then given by

$$\dot{\tilde{W}}_{2\{U_2\}}(x, t) = \begin{cases} [x_2 \quad x_1 \quad 0] [x_2 h(t) ; -x_1 - x_2 ; 1] & x \notin \overline{\text{Sq}} \wedge |x_1| \neq 1 \wedge |x_2| \neq 1 \\ & \forall x \in \text{Sq} \wedge |x_1| \neq |x_2|, \\ 0 & x \in \text{Sq} \wedge |x_1| = 0 \wedge |x_2| = 0, \\ -\infty & \text{otherwise.} \end{cases}$$

That is, $\dot{\tilde{W}}_{2\{U_2\}}(x, t) \leq -x_1^2 - x_2 x_1 + 2x_2^2, \forall (x, t) \in \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$. If $Y_2(z, x) := -x_1^2 - x_2 x_1 + 2x_2^2, \forall (z, x) \in \mathbb{R} \times \mathbb{R}^2$, then the functions $\{Y_1, Y_2\}$ have the Matrosov property. Furthermore, since $W_1, W_2 \in \mathcal{C}^0(\mathbb{R}^2 \times \mathbb{R}_{\geq 0}, \mathbb{R})$, $\forall 0 < \delta < \Delta, \exists \gamma > 0$ such that $|W(x, t)| \leq \gamma, \forall (x, t) \in D(\delta, \Delta) \times \mathbb{R}_{\geq 0}$. Hence, $\{W_1, W_2\}$ are \mathcal{U} -reduced Matrosov functions for $(F, \delta, \Delta), \forall 0 < \delta < \Delta$. Hence, by Thm. 8.10, (1) is uniformly globally asymptotically stable at $x = 0$.

9. CONCLUSION

This paper demonstrates that locally Lipschitz, regular functions can be used to identify infeasible directions in set-valued maps that define differential inclusions. The infeasible directions can then be removed to yield a point-wise smaller (in the sense of set containment) set-valued map that defines an equivalent differential inclusion. The reduction process results in a novel generalization of the set-valued derivative for locally Lipschitz candidate Lyapunov functions. Statements of Lyapunov stability theorems, invariance theorems, invariance-like results, and Matrosov theorems for differential inclusions that are less conservative than those available in the literature are developed using reduced set-valued maps.

The fact that arbitrary locally Lipschitz, regular functions can be used to restrict differential inclusions to smaller sets of admissible directions indicates that there may be a *smallest* set of admissible directions corresponding to each differential inclusion. Further research is needed to establish the existence of such a set and to find a representation of it that facilitates computation.

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