Time-Varying Input and State Delay Compensation for Uncertain Nonlinear Systems

Rushikesh Kamalapurkar, Nicholas Fischer, Serhat Obuz, and Warren E. Dixon

***Abstract*—A robust controller is developed for uncertain, second-order nonlinear systems subject to simultaneous unknown, time-varying state delays and known, sufficiently small time-varying input delays in addition to additive, sufficiently smooth disturbances. An integral term composed of previous control values facilitates a delay-free open-loop error system and the development of the feedback control structure. A stability analysis based on Lyapunov-Krasovskii (LK) functionals guarantees uniformly ultimately bounded tracking under the assumption that the delays are bounded and slowly varying. Numerical simulations illustrate robustness of the developed method to various combinations of simultaneous input and state delays.**

I. Introduction

Numerous control techniques exist for linear systems with constant input delays (cf. [1]–[3] and references therein). Many of these results are extensions of classic techniques such as Smith predictors [4], Artstein model reduction [5], or finite spectrum assignment [6]. Results that focus on simultaneous constant state and input delays for linear systems are provided in [7]–[9]. Extensions of linear control techniques to time-varying input delays are also available [10]–[15].

For nonlinear systems, controllers considering constant [16]–[23] and time-varying [20], [24]–[32] state delays have been recently developed. However, linear results considering delayed inputs are far less prevalent, especially for systems with model uncertainties and/or disturbances. Examples of these include constant input delay results in [33]–[45] and time-varying input delay results based on linear matrix inequality conditions [46], [47], backstepping [48], [49] and other robust techniques [50]. Even more unique are results that consider both state and input delays in nonlinear systems. Recently in [49], the predictor-based techniques in [7] were extended to nonlinear systems with time-varying delays in the state and/or the input utilizing a backstepping transformation to construct a predictor-based compensator. The development in [49] requires knowledge of the plant dynamics and assumes that the plant is disturbance-free.

In this paper, we expand our previous time-varying input delay result [50] in two directions: a) Utilizing techniques for constant input-delayed systems first introduced in [45], we consider time-varying input delays in a nonlinear plant, and b) we add the ability to compensate for simultaneous arbitrarily large unknown time-varying state delays based on the techniques in [32]. Robust control methods are developed to compensate for the unknown time-varying state delays. To compensate for the input delay, an integral term composed of previous control values is used to yield a delay-free open-loop system. A Lyapunov-based stability analysis motivated by Lyapunov-Krasovskii (LK) functionals demonstrates the ability to achieve uniformly ultimately bounded tracking in the presence of model uncertainty, additive sufficiently smooth disturbances, time-varying state delays and sufficiently small input delays. The result is based on the assumption that the unknown state delay is bounded and slowly varying. Improving on the result in [50], we relax previous sufficient conditions on the control gains that required knowledge of the second derivative of the input delay. Numerical simulations examine the robustness of the method to various combinations of simultaneous input and state delays.

II. Dynamic System

Consider a class of second-order[1] (Euler-Lagrange-like) nonlinear systems given by:[2]

$$\ddot{x} = f\left(x, \dot{x}, t\right) + g\left(x\left(t-\tau_s\right), \dot{x}\left(t-\tau_s\right), t\right) + d + u_{\tau_i}, \quad (1)$$

where $x, \dot{x} \in \mathbb{R}^n$ are the system states, $u \in \mathbb{R}^n$ is the control input, $f : \mathbb{R}^n \times \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^n$ is an unknown function, uniformly bounded in t, $g : \mathbb{R}^n \times \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^n$ is an unknown function with delayed internal state, uniformly bounded in t, $d \in \mathbb{R}^n$ denotes a sufficiently smooth disturbance (e.g., unmodeled effects), and $\tau_i, \tau_s \in [0,\infty)$ denote time-varying, non-negative input and state delays, respectively.

The subsequent development is based on the assumption that x and $\dot{x}$ are measurable outputs. Throughout the paper,

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[1] The result in this paper can be extended to n^{th}-order nonlinear systems following a similar development to that presented in [31].

[2] For brevity of notation, unless otherwise specified, the domain of all the functions is assumed to be $\mathbb{R}_{\geq 0}$. Furthermore, unless otherwise specified, all mathematical quantities are assumed to be time-varying and time-dependence is suppressed in equations and definitions. For example, the trajectory $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is defined by abuse of notation as $x \in \mathbb{R}^n$ and unless otherwise specified, an equation (inequality) of the form $f + h\left(y,t\right) = (\leq)\, g\left(x\right)$ is interpreted as $f\left(t\right) + h\left(y\left(t\right), t\right) = (\leq)\, g\left(x\left(t\right)\right)$ for all $t \in \mathbb{R}_{\geq 0}$.

a time-dependent delayed function is denoted as

$$\zeta_{\tau} \triangleq \begin{cases} \zeta\left(t-\tau\right) & t-\tau > t_0 \\ 0 & t-\tau \leq t_0, \end{cases}$$

where t_0 denotes the initial time. Thus, u_{τ_i} is defined by

$$u_{\tau_i} \triangleq \begin{cases} u\left(t-\tau_i\right) & t-\tau_i > t_0 \\ 0 & t-\tau_i \leq t_0. \end{cases}$$

Additionally, $\|\cdot\|$ denotes the Euclidean norm of a vector and the following assumptions will be exploited.

Assumption 1. Each of the functions f and g, along with their first and second partial derivatives, is bounded on each subset of its domain of the form $\mathscr{K} \times [0, \infty)$, where $\mathscr{K} \subset \mathbb{R}^n \times \mathbb{R}^n$ is compact. Furthermore, given such compact $\mathscr{K}$, the corresponding bound is known.

Assumption 2. The nonlinear disturbance term and its time derivative are bounded by known constants.[3]

Assumption 3. The desired trajectory $x_d \in \mathbb{R}^n$ is designed such that $x_d^{(i)} \in \mathbb{R}^n, \forall i = 0, 1, ..., 3$ exist and are bounded by known positive constants, where the superscript (i) denotes the i^{th} time derivative.[4]

Assumption 4. The input and state delays are bounded such that $0 \leq \tau_i \leq \varphi_{i_1}$ and $0 \leq \tau_s \leq \varphi_{s_1}$, and the rate of change of the delays are bounded such that $|\dot{\tau}_i| \leq \varphi_{i_2} < 1$ and $|\dot{\tau}_s| \leq \varphi_{s_2} < 1$ where $\varphi_j \in \mathbb{R}^+ \, \forall j = i_1, i_2, s_1, s_2$ are known constants. Furthermore, the bounds on the input delay satisfy $\varphi_{i_1} + \varphi_{i_2} < 1$. The state delay is assumed to be unknown, while the input delay is assumed to be known.

Remark 1. In Assumption 4, the slowly time-varying constraint (i.e., $|\dot{\tau}_{i,s}| \leq \varphi_{i_2,s_2} < 1$) is common to results which utilize classical LK functionals to compensate for time-varying time-delays [14]. Knowledge of the state delays in the system is not required; however, the input delays present a more significant challenge. Although the controller requires the input delay to be known so that the interval of past control values can be properly sized, numerical simulations illustrate robustness to uncertainties in the input delay.

III. Control Objective

The objective is to design a controller that will ensure the system state x of the system in (1) tracks a desired state trajectory. To quantify the control objective, a tracking error, denoted by $e_1 \in \mathbb{R}^n$, is defined as

$$e_1 \triangleq x_d - x. \tag{2}$$

To facilitate the subsequent analysis, two auxiliary tracking errors $e_2, r \in \mathbb{R}^n$ are defined as [45]

$$e_2 \triangleq \dot{e}_1 + \alpha_1 e_1, \tag{3}$$

$$r \triangleq \dot{e}_2 + \alpha_2 e_2 + e_u, \tag{4}$$

where $\alpha_1, \alpha_2 \in \mathbb{R}$ denote constant positive control gains, and $e_u \in \mathbb{R}^n$ denotes the mismatch between the delayed control input and the computed control input, defined as[5]

$$e_u \triangleq u_{\tau_i} - u. \tag{5}$$

The auxiliary signal e_u injects a delay-free control input into the error system development. In contrast to the development in [50], the term in (5) is embedded in a higher order derivative (i.e., r instead of e_2). Functionally, e_u still injects an integral of past control values into the open-loop system; however, the development introduces fewer cross-terms. The auxiliary signal r is introduced to facilitate the stability analysis and is not used in the control design since the expression in (4) depends on the unmeasurable state $\ddot{x}$. The structure of the error systems is motivated by the need to inject and cancel terms in the subsequent stability analysis as demonstrated in Section IV.

Using (1), (2), (3), and (5) to eliminate the delayed input term, (4) can be represented as

$$r = S_1 + S_2 - u, \tag{6}$$

where the auxiliary functions $S_1 \in \mathbb{R}^n$ and $S_2 \in \mathbb{R}^n$ are defined as

$$\begin{aligned} S_1 \triangleq\; & f\left(x_d, \dot{x}_d, t\right) - f\left(x, \dot{x}, t\right) + g\left(x_{d\tau_s}, \dot{x}_{d\tau_s}, t\right) \\ & - g\left(x_{\tau_s}, \dot{x}_{\tau_s}, t\right) + \alpha_1 \dot{e}_1 + \alpha_2 e_2, \\ S_2 \triangleq\; & \ddot{x}_d - f\left(x_d, \dot{x}_d, t\right) - g\left(x_{d\tau_s}, \dot{x}_{d\tau_s}, t\right) - d. \end{aligned}$$

Based on (6) and the subsequent stability analysis,

$$u \triangleq K\left(e_2 - e_2\left(t_0\right)\right) + v, \tag{7}$$

where $v \in \mathbb{R}^n$ is the solution to the following differential equation

$$\dot{v} = K\left(\alpha_2 e_2 + e_u\right), \; v\left(t_0\right) = 0, \tag{8}$$

and $K \in \mathbb{R}$ is a positive constant control gain. In contrast to the development in [50], the controller developed in (7) and (8) acts as a proportional-integral-derivative feedback in the absence of delay.

The closed-loop error system can be developed by taking the time derivative of (6) and substituting for (4) and the time derivative of (7) to yield

$$\dot{r} = \tilde{N} + N_d - e_2 - Kr, \tag{9}$$

where $\tilde{N} \in \mathbb{R}^n$ and $N_d \in \mathbb{R}^n$ are defined as

$$\tilde{N} \triangleq \dot{S}_1 + e_2, \tag{10}$$

$$N_d \triangleq \dot{S}_2. \tag{11}$$

The design of the auxiliary signal in (5) is motivated by the desire to eliminate the delayed input, yielding the closed-loop error system in (9). The structure of (9) is advantageous because it facilitates the stability analysis by segregating

[3]Many practical disturbance terms are continuous including friction (see [51], [52]), wind disturbances, wave/ocean disturbances, etc.

[4]Many guidance and navigation applications utilize smooth, high-order differentiable desired trajectories. Curve fitting methods can also be used to generate sufficiently smooth time-varying trajectories.

[5]Let $h \triangleq \max\left(t_0, t-\tau_i\right)$. Then, $h : [0, \infty) \to [0, \infty)$ is continuous. Further, since $u\left(t_0\right) = 0$, $u_{\tau_i} = u\left(h\right)$, and $e_u = u\left(h\right) - u$. Hence, e_u is a continuous function of time if u is a continuous function of time, and $e_u\left(t_0\right) = 0$.

terms that can be upper bounded by a state-dependent term and terms that can be upper bounded by constants. Based on Assumptions 2 and 3, the following inequalities can be developed from the expression in (11):

$$\|N_d\| \leq \zeta_{N_{d1}}, \tag{12}$$

where $\zeta_{N_{d1}} \in \mathbb{R}$, is a known positive constant. The Mean Value Theorem can be utilized to find an upper bound for the expression in (10) as [53, Lemma 2]

$$\left\|\tilde{N}\right\| \leq \rho_1\left(\|z\|\right)\|z\| + \rho_2\left(\|z_{\tau_s}\|\right)\|z_{\tau_s}\|, \tag{13}$$

where $z \in \mathbb{R}^{4n}$ denotes the vector

$$z \triangleq \left[e_1^T,\ e_2^T,\ r^T,\ e_u^T\right]^T, \tag{14}$$

and the bounding terms $\rho_1, \rho_2 : [0,\infty) \to [0,\infty)$ are positive, non-decreasing and radially unbounded functions.[6] The upper bound for the auxiliary function $\tilde{N}$ is segregated into delay-free and delay-dependent bounding functions to eliminate the delayed terms with the use of an LK term in the stability analysis.

To facilitate the subsequent stability analysis, several auxiliary terms are introduced. Let the control gain K be split in two parts as $K = k_s + k_{s1}$. Let $\{\iota_i \in \mathbb{R} \mid i = 1, \cdots, 8\}$ be positive adjustable constants such that $\iota_2 + \iota_3 + \iota_4 = 1$, and $\iota_5 + \iota_6 + \iota_7 + \iota_8 = 1$. Let $\rho : [0,\infty) \to [0,\infty)$ be an auxiliary bounding function defined as

$$\rho\left(\|z\|\right) = \sqrt{\left(\gamma_1 + 2\gamma_2\varphi_{s_1}\right)\rho_2^2\left(\|z\|\right) + \frac{\rho_1^2\left(\|z\|\right)}{\iota_3}}, \tag{15}$$

where γ_1 and γ_2 are positive adjustable constants, and let $\acute{z} \in \mathbb{R}^{3n}$ be defined as

$$\acute{z} \triangleq \left[e_1^T,\ e_2^T,\ r^T\right]^T. \tag{16}$$

Auxiliary bounding constants $\sigma, \delta \in \mathbb{R}$ are defined as

$$\sigma \triangleq \frac{1}{2}\min\left\{\frac{\alpha_1}{2}, \frac{\alpha_2}{2}, k_{s1}, \frac{\omega\iota_5\left(1-\varphi_{i_2}\right)}{\varphi_{i_1}}\right\}, \tag{17}$$

$$\delta \triangleq \frac{1}{2}\min\Bigg\{\sigma, \frac{\omega\iota_7\left(1-\varphi_{i_2}\right)}{\varphi_{i_1}}, \frac{\iota_6\left(1-\varphi_{i_2}\right)}{\varphi_{i_1}}, \frac{\gamma_2\left(1-\varphi_{s_2}\right)}{\gamma_1}, \frac{\left(1-\varphi_{s_2}\right)}{2\varphi_{s_1}}\Bigg\}, \tag{18}$$

where $\omega \in \mathbb{R}$ is a known, positive, adjustable constant.

Let

$$\mathscr{D} \triangleq \left\{\xi \in \mathbb{R}^{3n+4} \middle| \|\xi\| < \inf\left\{\rho^{-1}\left(\left[\sqrt{2k_s\sigma}, \infty\right)\right)\right\}\right\},$$

and

$$\mathcal{S}_{\mathscr{D}} \triangleq \left\{\xi \in \mathscr{D} \middle| \|\xi\| < \sqrt{\frac{1}{2}}\inf\left\{\rho^{-1}\left(\left[\sqrt{2k_s\sigma}, \infty\right)\right)\right\}\right\},$$

where, for a set $A \subset \mathbb{R}$, the inverse image $\rho^{-1}(A) \subset \mathbb{R}$ is defined as $\rho^{-1}(A) \triangleq \{a \in \mathbb{R} \mid \rho(a) \in A\}$. Furthermore, let the functions[7] $P_{LK} : [0,\infty) \to [0,\infty)$, $Q_{LK} : [0,\infty) \to [0,\infty)$, $R_{LK} : [0,\infty) \to [0,\infty)$, and $S_{LK} : [0,\infty) \to [0,\infty)$ be defined as

$$P_{LK} \triangleq \varphi_{i_1}\int_{t-\tau_i}^{t}\|\dot{u}(\theta)\|^2 d\theta, \tag{19}$$

$$Q_{LK} \triangleq \omega\int_{t-\tau_i}^{t}\left(\int_{s}^{t}\|\dot{u}(\theta)\|^2 d\theta\right)ds, \tag{20}$$

$$R_{LK} \triangleq \frac{\gamma_1}{2k_s}\int_{t-\tau_s}^{t}\rho_2^2\left(\|z(\sigma)\|\right)\|z(\sigma)\|^2 d\sigma, \tag{21}$$

$$S_{LK} \triangleq \frac{\gamma_2}{k_s}\int_{t-\tau_s}^{t}\left(\int_{s}^{t}\rho_2^2\left(\|z(\sigma)\|\right)\|z(\sigma)\|^2 d\sigma\right)ds. \tag{22}$$

Additionally, let $y \in \mathbb{R}^{3n+4}$ be defined as

$$y \triangleq \left[\begin{array}{ccccc}\acute{z}^T & \sqrt{P_{LK}} & \sqrt{Q_{LK}} & \sqrt{R_{LK}} & \sqrt{S_{LK}}\end{array}\right]^T. \tag{23}$$

IV. Stability Analysis

Theorem 1. *Given the dynamics in (1), provided the control gains are selected based on the following sufficient conditions*

$$\alpha_1 > 1,\ \alpha_2 > \left(1+\iota_1\right),\ \gamma_1 > \frac{1}{\left(1-\varphi_{s_2}\right)}, \quad \omega > \frac{\varphi_{i_1}}{2\left(1-\varphi_{i_2}\right)\iota_1\iota_8}, \tag{24}$$

and the input delay is small enough so that there exists a gain k_s that satisfies the sufficient conditions[8]

$$\varphi_{i_1} < \frac{\iota_2 k_s}{2\left(\omega+1\right)K^2}, \tag{25}$$

$$\frac{\zeta_{N_{d1}}^2}{\iota_4 k_s\delta} < \left(\inf\left\{\rho^{-1}\left(\left[\sqrt{2k_s\sigma}, \infty\right)\right)\right\}\right)^2, \tag{26}$$

the controller given in (7) and (8) ensures uniformly ultimately bounded tracking in the sense that $\limsup_{t\to\infty}\|y\| \leq \sqrt{\frac{\zeta_{N_{d1}}^2}{\iota_4 k_s\delta}}$, *provided* $y(t_0) \in \mathcal{S}_{\mathscr{D}}$.

Proof: Let $V : \mathscr{D} \to \mathbb{R}$ be a candidate Lyapunov function defined as

$$V \triangleq \frac{1}{2}e_1^T e_1 + \frac{1}{2}e_2^T e_2 + \frac{1}{2}r^T r + P_{LK} + Q_{LK} + R_{LK} + S_{LK}, \tag{27}$$

which satisfies the following inequalities:

$$\frac{1}{2}\|y\|^2 \leq V(y) \leq \|y\|^2. \tag{28}$$

The time derivative of (27) can be found by applying the Leibniz Rule to (19), (20), (21) and (22), and by substituting (2)-(4), (7), and (9), yielding

$$\begin{aligned}\dot{V} &= e_1^T\left(e_2 - \alpha_1 e_1\right) + e_2^T\left(r - \alpha_2 e_2 - e_u\right) \\ &\quad + r^T\left(\tilde{N} + N_d - e_2 - Kr\right) \\ &\quad + \left(\omega\tau_i + \varphi_{i_1}\right)K^2\|r\|^2 - \varphi_{i_1}\left(1-\dot{\tau}_i\right)\|\dot{u}_{\tau_i}\|^2 \\ &\quad - \omega\left(1-\dot{\tau}_i\right)\int_{t-\tau_i}^{t}\|\dot{u}(\theta)\|^2 d\theta\end{aligned}$$

[6]For some classes of systems, the bounding functions ρ_1 and ρ_2 could be selected as constants. For these classes of systems, a global uniformly ultimately bounded result can be obtained as described in Remark 2.

[7]The construction of P_{LK}, Q_{LK}, R_{LK}, and S_{LK} is based on LK functionals. However, in this result, they are to be interpreted as time-varying signals that are a part of the system state.

[8]Since δ increases with increasing k_s, the left-hand side of (26) decreases with increasing k_s. Since ρ is a nondecreasing function, the right-hand side of (26) is nondecreasing with respect to k_s. Hence, (26) can be satisfied for some k_s. Furthermore, for any given k_s, (25) is satisfied if the delay is small enough.

$$
\begin{aligned}
&+\left(\frac{\gamma_1}{2k_s}+\frac{\gamma_2}{k_s}\tau_s\right)\rho_2^2\left(\|z\|\right)\|z\|^2\\
&-\frac{\gamma_1\left(1-\dot{\tau}_s\right)}{2k_s}\rho_2^2\left(\|z_{\tau_s}\|\right)\|z_{\tau_s}\|^2\\
&-\frac{\gamma_2}{k_s}\left(1-\dot{\tau}_s\right)\int_{t-\tau_s}^{t}\rho_2^2\left(\|z\left(\theta\right)\|\right)\|z\left(\theta\right)\|^2d\theta. \qquad (29)
\end{aligned}
$$

Using (4), (12), (13), the inequality $\dot{\tau}_s<1$ and Young's Inequality to show that $\left\|e_1^Te_2\right\|\leq\frac{1}{2}\|e_1\|^2+\frac{1}{2}\|e_2\|^2$, $\left\|e_2^Te_u\right\|\leq\frac{\iota_1}{2}\|e_2\|^2+\frac{1}{2\iota_1}\|e_u\|^2$ and $\|r\|\rho_2\left(\|z_{\tau_s}\|\right)\|z_{\tau_s}\|\leq\frac{k_s}{2}\|r\|^2+\frac{1}{2k_s}\rho_2^2\left(\|z_{\tau_s}\|\right)\|z_{\tau_s}\|^2$, the expression in (29) can be upper bounded as

$$
\begin{aligned}
\dot{V}\leq&-\alpha_1\|e_1\|^2-\alpha_2\|e_2\|^2-\left(\frac{k_s}{2}+k_{s1}\right)\|r\|^2\\
&+\frac{1}{2}\|e_1\|^2+\left(\frac{1}{2}+\frac{\iota_1}{2}\right)\|e_2\|^2+\frac{1}{2\iota_1}\|e_u\|^2\\
&+\frac{1}{2k_s}\rho_2^2\left(\|z_{\tau_s}\|\right)\|z_{\tau_s}\|^2-\frac{\gamma_1\left(1-\dot{\tau}_s\right)}{2k_s}\rho_2^2\left(\|z_{\tau_s}\|\right)\|z_{\tau_s}\|^2\\
&+\varphi_{i_1}\left(1+\omega\right)K^2\|r\|^2+\|r\|\zeta_{N_{d1}}\\
&-\omega\left(1-\dot{\tau}_i\right)\int_{t-\tau_i}^{t}\|\dot{u}\left(\theta\right)\|^2d\theta+\|r\|\rho_1\left(\|z\|\right)\|z\|\\
&+\left(\frac{\gamma_1}{2k_s}+\frac{\gamma_2}{k_s}\tau_s\right)\rho_2^2\left(\|z\|\right)\|z\|^2\\
&-\frac{\gamma_2}{k_s}\left(1-\dot{\tau}_s\right)\int_{t-\tau_s}^{t}\rho_2^2\left(\|z\left(\theta\right)\|\right)\|z\left(\theta\right)\|^2d\theta. \qquad (30)
\end{aligned}
$$

Using the fact that $e_u=\int_{t-\tau(t)}^{t}\dot{u}\left(\theta\right)d\theta$, the Cauchy-Schwarz inequality can be used to establish the bound $\|e_u\|^2\leq\tau_i\int_{t-\tau_i}^{t}\|\dot{u}\left(\theta\right)\|^2d\theta$. Completing the squares for r, utilizing the inequalities

$$
-\int_{t-\tau_i}^{t}\|\dot{u}\left(\theta\right)\|^2d\theta\leq-\frac{1}{\tau_i}\int_{t-\tau_i}^{t}\left(\int_{s}^{t}\|\dot{u}\left(\theta\right)\|^2d\theta\right)ds=-\frac{Q_{LK}}{\omega\tau_i},
$$

and

$$
\begin{aligned}
&-\int_{t-\tau_s}^{t}\rho_2^2\left(\|z\left(\theta\right)\|\right)\|z\left(\theta\right)\|^2d\theta\leq\\
&-\frac{1}{\tau_s}\int_{t-\tau_s}^{t}\left(\int_{s}^{t}\rho_2^2\left(\|z\left(\theta\right)\|\right)\|z\left(\theta\right)\|^2d\theta\right)ds=-\frac{k_sS_{LK}}{\gamma_2\tau_s},
\end{aligned}
$$

and (19), (21), and (26), (30) can be rewritten as

$$
\begin{aligned}
\dot{V}\leq&-\frac{\alpha_1}{2}\|e_1\|^2-\frac{\alpha_2}{2}\|e_2\|^2-k_{s1}\|r\|^2-\frac{\omega\iota_5(1-\dot{\tau}_i)}{\tau_i}\|e_u\|^2\\
&-\left(\frac{\alpha_1}{2}-\frac{1}{2}\right)\|e_1\|^2-\left(\frac{\alpha_2}{2}-\frac{1}{2}-\frac{\iota_1}{2}\right)\|e_2\|^2\\
&-\left(\frac{\iota_2k_s}{2}-\varphi_{i_1}\left(1+\omega\right)K^2\right)\|r\|^2+\frac{\zeta_{N_{d1}}^2}{2\iota_4k_s}\\
&-\left(\frac{\omega\iota_8\left(1-\dot{\tau}_i\right)}{\tau_i}-\frac{1}{2\iota_1}\right)\|e_u\|^2-\frac{\left(1-\dot{\tau}_s\right)}{2\tau_s}S_{LK}\\
&-\left(\frac{\gamma_1\left(1-\dot{\tau}_s\right)}{2k_s}-\frac{1}{2k_s}\right)\rho_2^2\left(\|z_{\tau_s}\|\right)\|z_{\tau_s}\|^2\\
&+\frac{1}{2k_s}\left(\frac{\rho_1^2\left(\|z\|\right)}{\iota_3}+\left(\gamma_1+2\gamma_2\varphi_{s_1}\right)\rho_2^2\left(\|z\|\right)\right)\|z\|^2\\
&-\frac{\omega\iota_7(1-\dot{\tau}_i)}{\varphi_{i_1}}P_{LK}-\frac{\iota_6(1-\dot{\tau}_i)}{\tau_i}Q_{LK}-\frac{\gamma_2(1-\dot{\tau}_s)}{\gamma_1}R_{LK}. \qquad (31)
\end{aligned}
$$

If the conditions in (24) are satisfied, based on the inequalities $\|z\|^2\geq\|\acute{z}\|^2$ and $\|z\|\leq\|y\|$, the expression in (31) reduces to

$$
\begin{aligned}
\dot{V}\leq&-\sigma\|\acute{z}\|^2-\frac{\omega\iota_7\left(1-\varphi_{i_2}\right)}{\varphi_{i_1}}P_{LK}-\frac{\iota_6\left(1-\varphi_{i_2}\right)}{\varphi_{i_1}}Q_{LK}\\
&-\frac{\gamma_2\left(1-\varphi_{s_2}\right)}{\gamma_1}R_{LK}-\frac{\left(1-\varphi_{s_2}\right)}{2\varphi_{s_1}}S_{LK}+\frac{\zeta_{N_{d1}}^2}{2\iota_4k_s},\\
\leq&-\delta\|y\|^2,\ \forall\|y\|\geq\sqrt{\frac{\zeta_{N_{d1}}^2}{2\iota_4k_s\delta}}, \qquad (32)
\end{aligned}
$$

provided $y\in\mathscr{D}$, where $\rho\left(\|z\|\right)$, σ, and δ were introduced in (15), (17) and (18). Using (26), (28), and (32), Theorem 4.18 in [54] can be invoked to conclude that y is uniformly ultimately bounded in the sense that $\limsup_{t\to\infty}\|y\|\leq\sqrt{\frac{\zeta_{N_{d1}}^2}{\iota_4k_s\delta}}$, provided $y\left(t_0\right)\in\mathcal{S}_{\mathscr{D}}$.

Since $e_1,e_2,r\in\mathcal{L}_\infty$, from (6), $u\in\mathcal{L}_\infty$, which implies $u_{\tau_i}\in\mathcal{L}_\infty$, and hence, $e_u\in\mathcal{L}_\infty$. The closed-loop error system can then be used to conclude that the remaining signals are bounded. ■

Remark 2. If the system dynamics are such that $\left\|\tilde{N}\right\|$ is linear in $\|z\|$, then the function ρ can be selected to be a constant, i.e., $\rho\left(\|z\|\right)=\overline{\rho}$, $\forall z\in\mathbb{R}^{4n}$ for some known $\overline{\rho}>0$. In this case, the gain condition in (26) reduces to $k_s>\frac{\overline{\rho}}{2\sigma}$, and the result is global in the sense that $\mathscr{D}=\mathcal{S}_{\mathscr{D}}=\mathbb{R}^{3n+4}$.

V. Simulation Results

The controller in (7) and (8) was simulated to examine the performance and robustness to variations in both the state and input delay. Specifically the dynamics[9] from (1) are utilized where $n=2$, $f\triangleq\begin{bmatrix}-p_4s_2\\p_5s_2\dot{x}_2\end{bmatrix}$, $g\triangleq\begin{bmatrix}-p_3s_{2\tau_s}\dot{x}_{2\tau_s} & -p_3s_{2\tau_s}\left(\dot{x}_{1\tau_s}+\dot{x}_{2\tau_s}\right)\\p_3s_{2\tau_s}\dot{x}_{1\tau_s} & 0\end{bmatrix}\begin{bmatrix}\dot{x}_{1\tau_s}\\\dot{x}_{2\tau_s}\end{bmatrix}+\begin{bmatrix}f_{d1} & 0\\0 & f_{d2}\end{bmatrix}\begin{bmatrix}\dot{x}_{1\tau_s}\\\dot{x}_{2\tau_s}\end{bmatrix}$, and $x,\dot{x},\ddot{x}\in\mathbb{R}^2$. A disturbance $d=\begin{bmatrix}d_1\\d_1\end{bmatrix}$ was applied where $d_1=0.2\sin\left(\frac{t}{2}\right)$ and $d_2=0.1\sin\left(\frac{t}{4}\right)$. Additionally, $p_1=3.473$, $p_2=0.196$, $p_3=0.242$, $p_4=0.238$, $p_5=0.146$, $f_{d1}=5.3$, $f_{d2}=1,1$, and s_2, $s_{2\tau_s}$ denote $\sin\left(x_2\right)$ and $\sin\left(x_2\left(t-\tau_s\right)\right)$.

The initial conditions for the system are selected as $x_1,x_2=0$. The desired trajectories are selected as

$$
\begin{aligned}
x_{d1}&=\left(30\sin\left(1.5t\right)+20\right)\left(1-e^{-0.01t^3}\right),\\
x_{d2}&=-\left(20\sin\left(t/2\right)+10\right)\left(1-e^{-0.01t^3}\right).
\end{aligned}
$$

To illustrate robustness to the delays, several simulations are completed using various time-varying delays. Various

[9] The expressions for f and g are derived from the expression for a two-link revolute, direct drive robot, which is a commonly used example Euler-Lagrange dynamic system.

Table I
RMS ERRORS FOR TIME-VARYING TIME-DELAY RATES AND MAGNITUDES.

	τ_s (ms)	τ_i (ms)	RMS Error x_1	RMS Error x_2
Case 1	$5sin\left(\frac{t}{8}\right)+10$	$-50sin\left(\frac{t}{10}\right)+100$	1.42^o	1.36^o
Case 2	$10sin\left(\frac{t}{2}\right)+40$	$-10sin\left(\frac{t}{3}\right)+30$	0.32^o	0.36^o
Case 3	$10sin\left(\frac{t}{10}\right)+40$	$-50sin\left(\frac{t}{2}\right)+100$	1.15^o	1.22^o
Case 4	$10sin\left(\frac{t}{10}\right)+40$	$50sin\left(\frac{t}{10}\right)+100$	1.54^o	1.83^o
Case 5	$50sin\left(t\right)+800$	$5sin\left(\frac{t}{2}\right)+30$	0.23^o	0.37^o

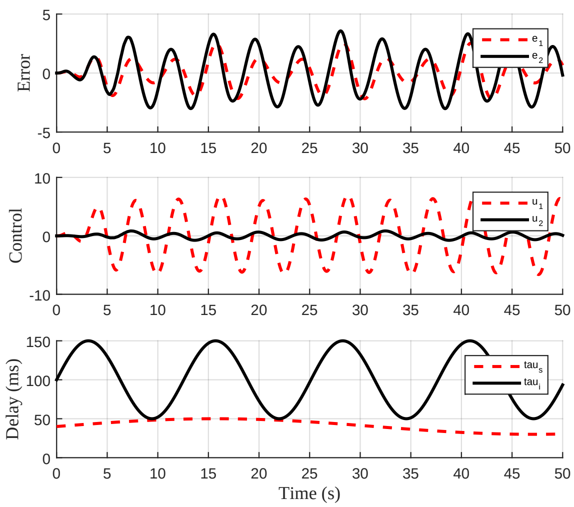

Figure 1. Tracking errors, actuation effort and time-varying delays vs time for Case 3.

time delay combinations are considered, and for each case the root mean square (RMS) errors are shown in Table I.[10] To illustrate the findings, Figure 1 depicts the tracking errors, actuation effort and time-varying delays for Case 3 in Table I. Additionally, Case 5 in Table I is provided in Figure 2. The results indicate that the performance of the system is relatively less sensitive to the delay frequency and more sensitive to the delay magnitude. This outcome agrees with previous input delay results where the tracking performance degrades as larger delays are applied to the system [43], [50]. In general, the simulation results illustrate that the proposed controller is able to achieve better tracking performance as well as tolerate larger input delays (even with added simultaneous state delays) than the time-varying input-delayed work in [50]. The controller is more robust to larger magnitude delays in the state than in the input, as indicated in the stability analysis and as demonstrated in Case 5.

VI. Conclusion

This paper presents a robust controller for uncertain nonlinear systems which include simultaneous time-varying state and input delays, as well as sufficiently smooth additive bounded disturbances. The controller utilizes a robust design approach to compensate for the unknown state delays coupled with an error system structure that provides a delay-free open-loop error system. The controller and LK functionals guarantee uniformly ultimately bounded tracking provided the rates of the delays are sufficiently slow. The controller can be applied when there is uncertainty in the system dynamics and when the state delay is unknown; however, the controller is based on the assumption that the time-varying input delay is known. Numerical simulations examine the robustness of the method to various combinations of simultaneous input and state delays. The simulation results illustrate robustness to uncertainty in the magnitude and frequency of the input delay and the state delay. These results point to the possibility that different control or analysis methods could be developed to eliminate the assumption that the input delay is known. That is, perhaps the interval of previous control values could somehow be designed big enough to provide predictive properties despite uncertainty in the input delay.

[10]The simulation results indicate that the condition $|\dot{\tau}_i| \leq \varphi_{i_2} < 1$ in Assumption 4 is sufficient in the sense that the developed controller can compensate for fast variations in the time delay.

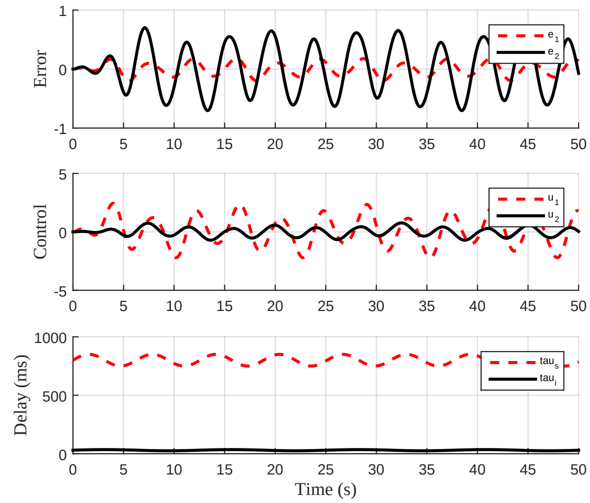

Figure 2. Tracking errors, actuation effort and time-varying delays vs time for Case 5.

References

[1] M. Krstic, *Delay Compensation for Nonlinear, Adaptive, and PDE Systems*. Springer, 2009.
[2] J. Chiasson and J. Loiseau, *Applications of time delay systems*, ser. Lecture notes in control and information sciences. Springer, 2007.
[3] K. Gu, V. L. Kharitonov, and J. Chen, *Stability of Time-delay systems*. Birkhauser, 2003.
[4] O. M. Smith, “A controller to overcome deadtime,” *ISA J.*, vol. 6, pp. 28–33, 1959.
[5] Z. Artstein, “Linear systems with delayed controls: A reduction,” *IEEE Trans. Autom. Control*, vol. 27, no. 4, pp. 869–879, 1982.
[6] A. Manitius and A. Olbrot, “Finite spectrum assignment problem for systems with delays,” *IEEE Trans. Autom. Control*, vol. 24, no. 4, pp. 541–552, 1979.
[7] N. Bekiaris-Liberis and M. Krstic, “Stabilization of linear strict feedback systems with delayed integrators,” *Automatica*, vol. 46, pp. 1902–1910, 2010.
[8] M. Jankovic, “Forwarding, backstepping, and finite spectrum assignment for time delay systems,” *Automatica*, vol. 45, pp. 2–9, 2009.
[9] ——, “Recursive predictor design for state and output feedback controllers for linear time delay systems,” *Automatica*, vol. 46, pp. 510–517, 2010.

[10] M. Nihtila, "Adaptive control of a continuous-time system with time-varying input delay," *IEEE Trans. Autom. Control*, vol. 12, pp. 357–364, 1989.

[11] R. Lozano, P. Castillo, P. Garcia, and A. Dzul, "Robust prediction-based control for unstable delay systems: Application to the yaw control of a mini-helicopter," *Automatica*, vol. 40, no. 4, pp. 603–612, 2004.

[12] D. Yue and Q.-L. Han, "Delayed feedback control of uncertain systems with time-varying input delay," *Automatica*, vol. 41, no. 2, pp. 233–240, 2005.

[13] Z. Wang, P. Goldsmith, and D. Tan, "Improvement on robust control of uncertain systems with time-varying input delays," *IET Control Theory Appl.*, vol. 1, no. 1, pp. 189 –194, 2007.

[14] J.-P. Richard, "Time-delay systems: an overview of some recent advances and open problems," *Automatica*, vol. 39, no. 10, pp. 1667 – 1694, 2003.

[15] M. Krstic, "Lyapunov stability of linear predictor feedback for time-varying input delay," *IEEE Trans. Autom. Control*, vol. 55, pp. 554–559, 2010.

[16] S. S. Ge, F. Hong, and T. H. Lee, "Adaptive neural control of nonlinear time-delay systems with unknown virtual control coefficients," *IEEE Trans. Syst. Man Cybern. Part B Cybern.*, vol. 34, no. 1, pp. 499–516, 2004.

[17] S. Ge, F. Hong, and T. Lee, "Robust adaptive control of nonlinear systems with unknown time delays," *Automatica*, vol. 41, no. 7, pp. 1181–1190, Jul. 2005.

[18] S. J. Yoo, J. B. Park, and C. H. Choi, "Adaptive dynamic surface control for stabilization of parametric strict-feedback nonlinear systems with unknown time delays," *IEEE Trans. Autom. Control*, vol. 52, no. 12, pp. 2360–2365, 2007.

[19] C.-C. Hua, X.-P. Guan, and G. Feng, "Robust stabilisation for a class of time-delay systems with triangular structure," *IET Control Theory Appl.*, vol. 1, no. 4, pp. 875–879, 2007.

[20] H. Wu, "Adaptive robust state observers for a class of uncertain nonlinear dynamical systems with delayed state perturbations," *IEEE Trans. Autom. Control*, vol. 54, no. 6, pp. 1407–1412, 2009.

[21] M. Wang, B. Chen, and S. Zhang, "Adaptive neural tracking control of nonlinear time-delay systems with disturbances," *Int. J. Adapt Control Signal Process.*, vol. 23, pp. 1031–1049, 2009.

[22] S.-C. Tong and N. Sheng, "Adaptive fuzzy observer backstepping control for a class of uncertain nonlinear systems with unknown time-delay," *Int.. J. Autom. and Comput.*, vol. 7, no. 2, pp. 236–246, 2010.

[23] A. Kuperman and Q.-C. Zhong, "Robust control of uncertain nonlinear systems with state delays based on an uncertainty and disturbance estimator," *Int. J. Robust Nonlinear Control*, vol. 21, pp. 79–92, 2011.

[24] H. Huang and D. Ho, "Delay-dependent robust control of uncertain stochastic fuzzy systems with time-varying delay," *IET Control Theory Appl.*, vol. 1, no. 4, pp. 1075–1085, 2007.

[25] B. Ren, S. S. Ge, T. H. Lee, and C. Su, "Adaptive neural control for a class of nonlinear systems with uncertain hysteresis inputs and time-varying state delays," *IEEE Trans. Neural Netw.*, vol. 20, pp. 1148–1164, 2009.

[26] S. J. Yoo and J. B. Park, "Neural-network-based decentralized adaptive control for a class of large-scale nonlinear systems with unknown time-varying delays," *IEEE Trans. Syst. Man Cybern.*, vol. 39, no. 5, pp. 1316–1323, 2009.

[27] M. Wang, S. S. Ge, and K. Hong, "Approximation-based adaptive tracking control of pure-feedback nonlinear systems with multiple unknown time-varying delays," *IEEE Trans. Neur. Netw.*, vol. 21, no. 11, pp. 1804–1816, 2010.

[28] Y. Niu, D. W. C. Ho, and J. Lam, "Robust integral sliding mode control for uncertain stochastic systems with time-varying delay," *Automatica*, vol. 41, pp. 873–880, 2005.

[29] W. Chen, L. Jiao, J. Li, and R. Li, "Adaptive nn backstepping output-feedback control for stochastic nonlinear strict-feedback systems with time-varying delays," *IEEE Trans. Syst. Man Cybern.*, vol. 40, no. 3, pp. 939–950, 2010.

[30] B. Mirkin and P.-O. Gutman, "Robust adaptive output-feedback tracking for a class of nonlinear time-delayed plants," *IEEE Trans. Autom. Control*, vol. 55, no. 10, pp. 2418–2424, 2010.

[31] N. Sharma, S. Bhasin, Q. Wang, and W. E. Dixon, "RISE-based adaptive control of a control affine uncertain nonlinear system with unknown state delays," *IEEE Trans. Autom. Control*, vol. 57, no. 1, pp. 255–259, Jan. 2012.

[32] N. Fischer, R. Kamalapurkar, N. Sharma, and W. E. Dixon, "RISE-based control of an uncertain nonlinear system with time-varying state delays," in *Proc. IEEE Conf. Decis. Control*, Maui, HI, Dec. 2012, pp. 3502–3507.

[33] F. Mazenc and S.-I. Niculescu, "Generating positive and stable solutions through delayed state feedback," *Automatica*, vol. 47, pp. 525–533, 2011.

[34] I. Karafyllis, "Stabilization by means of approximate predictors for systems with delayed input," *SIAM J. Control Optim.*, vol. 49, no. 3, pp. 1100–1123, 2011.

[35] M. Krstic and A. Smyshlyaev, "Backstepping boundary control for first-order hyperbolic PDEs and application to systems with actuator and sensor delays," *Syst. Control Lett.*, vol. 57, no. 9, pp. 750–758, 2008.

[36] D. Bresch-Pietri and M. Krstic, "Adaptive trajectory tracking despite unknown input delay and plant parameters," *Automatica*, vol. 45, no. 9, pp. 2074–2081, 2009.

[37] F. Mazenc, S. Mondie, R. Francisco, P. Conge, I. Lorraine, and F. Metz, "Global asymptotic stabilization of feedforward systems with delay in the input," *IEEE Trans. Autom. Control*, vol. 49, (5), pp. 844–850, 2004.

[38] F. Carravetta, P. Palumbo, and P. Pepe, "Quadratic optimal control of linear systems with time-varying input delay," in *Proc. IEEE Conf. Decis. Control*, 2010, pp. 4996–5000.

[39] B. Chen, X. Liu, and S. Tong, "Robust fuzzy control of nonlinear systems with input delay," *Chaos, Solitons & Fractals*, vol. 37, no. 3, pp. 894–901, 2008.

[40] F. Mazenc, S. Niculescu, and M. Bekaik, "Stabilization of time-varying nonlinear systems with distributed input delay by feedback of plant's state," *IEEE Trans. Autom. Control*, vol. PP, no. 99, p. 1, 2012.

[41] P. Pepe, Z.-P. Jiang, and E. Fridman, "A new Lyapunov-Krasovskii methodology for coupled delay differential and difference equations," *Int. J. Control*, vol. 81, pp. 107–115, 2008.

[42] M. Krstic, "Input delay compensation for forward complete and strict-feedforward nonlinear systems," *IEEE Trans. Autom. Control*, vol. 55, pp. 287–303, Feb. 2010.

[43] N. Sharma, S. Bhasin, Q. Wang, and W. E. Dixon, "Predictor-based control for an uncertain Euler-Lagrange system with input delay," *Automatica*, vol. 47, no. 11, pp. 2332–2342, 2011.

[44] B. Castillo-Toledo, S. Di Gennaro, and G. Castro, "Stability analisys for a class of sampled nonlinear systems with time-delay," in *Proc. IEEE Conf. Decis. Control*, 2010, pp. 1575 –1580.

[45] S. Obuz, E. Tatlicioglu, S. C. Cekic, and D. M. Dawson, "Predictor-based robust control of uncertain nonlinear systems subject to input delay," in *IFAC Workshop on Time Delay Syst.*, vol. 10, no. 1, 2012, pp. 231–236.

[46] X. Jiao, J. Yang, and Q. Li, "Adaptive control for a class of nonlinear systems with time-varying delays in the state and input," *J. Control Theory Appl.*, vol. 9, pp. 183–188, 2011.

[47] J. Liu, J. Zhang, Y. Zheng, and M. He, "Robust H-infinity control for discrete-time T-S fuzzy systems with input delay," *J. Control Theory Appl.*, vol. 9, pp. 189–194, 2011.

[48] I. Karafyllis, "Finite-time global stabilization by means of time-varying distributed delay feedback," *SIAM J. Control Optim.*, vol. 45, pp. 320–342, 2006.

[49] N. Bekiaris-Liberis and M. Krstic, "Compensation of time-varying input and state delays for nonlinear systems," *J. Dyn. Syst. Meas. Control*, vol. 134, no. 1, p. 011009, 2012.

[50] N. Fischer, R. Kamalapurkar, N. Fitz-Coy, and W. E. Dixon, "Lyapunov-based control of an uncertain Euler-Lagrange system with time-varying input delay," in *Proc. Am. Control Conf.*, Montréal, Canada, June 2012, pp. 3919–3924.

[51] C. Makkar, G. Hu, W. G. Sawyer, and W. E. Dixon, "Lyapunov-based tracking control in the presence of uncertain nonlinear parameterizable friction," *IEEE Trans. Autom. Control*, vol. 52, pp. 1988–1994, 2007.

[52] C. Makkar, W. E. Dixon, W. G. Sawyer, and G.H, "A new continuously differentiable friction model for control systems design," in *Proc. IEEE/ASME Int. Conf. Adv. Intell. Mechatron.*, Monterey, CA, July 2005, pp. 600–605.

[53] R. Kamalapurkar, J. A. Rosenfeld, J. Klotz, R. J. Downey, and W. E. Dixon. (2014) Supporting lemmas for RISE-based control methods. arXiv:1306.3432.

[54] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Upper Saddle River, NJ, USA: Prentice Hall, 2002.