State and Parameter Estimation for Affine Nonlinear Systems

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***Abstract*— This paper proposes a new approach to online state and parameter estimation for affine nonlinear systems. Unlike conventional methods limited to specific classes of nonlinear systems and reliant on stringent excitation conditions, the proposed approach uses multiplier matrices and a data-driven concurrent learning method to develop an adaptive observer for affine nonlinear systems. Through rigorous Lyapunov-based analysis, the technique is proven to guarantee locally uniformly ultimately bounded state estimates and ultimately bounded parameter estimation errors. Additionally, under certain excitation conditions, the parameter estimation error is guaranteed to converge to a given neighborhood of the origin.**

I. Introduction

In many real-world control systems, the limited availability of sensor information and unknown model parameters make effective control of the system difficult, if not impossible. While adaptive control methods typically rely on full state measurement to generate parameter estimates, techniques that simultaneously estimate the system states and parameters are also available for nonlinear systems, albeit for specific classes of nonlinear systems [1]–[3]. This motivates the need for nonlinear observer techniques for simultaneous state and parameter estimation for a broader class of nonlinear systems.

In nonlinear state observers like the extended Luenberger nonlinear observers in [4]–[8], restricted to a specific class of nonlinear systems, incremental multiplier matrices are employed to characterize the nonlinearities in the system dynamics, and observer gain matrices are then obtained by solving linear matrix inequalities [7]–[9] using semi-definite programming. The drawback of extended Luenberger nonlinear observers is the need to compute bounds on the Jacobian matrices of unknown vector fields that model the system. Methods such as [10] and [11] offer solutions to challenges in calculating such Jacobian bounds and suitable multiplier matrices. However, due to their separation of measurable and unmeasurable signals and reliance on convex optimization techniques to formulate an explicit matrix polynomial form of the gradient, methods such as [10] and [11] are difficult to apply for simultaneous state and parameter estimation in nonlinear systems. Notwithstanding, the convergence properties of extended Luenberger state observers can be leveraged to generate precise state estimates for parameter estimation, even when only partial state measurements are available from the output [9].

Parameter estimation methods that rely on persistent excitation (PE) [12]–[14] and finite excitation [15]–[18] have also been studied extensively in the literature for systems where all state variables can be measured. Recent research efforts have focused on developing adaptive observers that can simultaneously estimate the state and parameters of nonlinear systems [1]–[3], [19]–[23]. However, most of these methods are also restricted to a specific class of nonlinear systems and rely on assumptions that may be difficult to satisfy in practice, such as stringent PE conditions [1]–[3], [22], [23]. Methods such as those developed in [1] and [3], while effective, are restricted to dynamical systems that are of the Brunovsky canonical form. Similarly, adaptive observers that use dynamic regressor extension and mixing (DREM) rely on the existence of a cascade form via a coordinate change for which a linear regression relation exists between the system states and unknown parameters [22].

Unlike the existing simultaneous state and parameter estimation methods described above, limited to narrow class systems, such as systems in Brunovsky form or cascade form, this paper presents a novel method that achieves simultaneous state and parameter estimation for a broader class of nonlinear systems. The key idea is to leverage the advantages of the multiplier matrix approach for Luenberger observer design, which has been proven to yield asymptotic convergence of state estimation errors [7], [9], and build upon concurrent learning (CL) frameworks [1], [3], which utilizes recorded data (stored in what is commonly called a history stack) to estimate parameters with high accuracy. In contrast with methods proposed in results such as [24]–[26], the developed method does not require any restrictions on the form and rank of the C measurement matrix or impose observability conditions.

The rest of the paper is organized as follows: Section II contains the problem formulation, Section III presents the state observer design, Section IV presents the parameter estimator design, Section V contains stability analysis of the developed method, and Section VII concludes the paper.

II. Problem Formulation

Consider a nonlinear dynamical system of the form

$$\dot{x} = Y(x)\theta + g(x)u, \quad y = Cx, \tag{1}$$

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where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ denotes the system state and the control input respectively, $\theta \in \Theta \subset \mathbb{R}^p$ is a vector of unknown parameters, $C \in \mathbb{R}^{q\times n}$ is the output matrix, and $y \in \mathbb{R}^q$ is the measured output. The functions $Y : \mathbb{R}^n \to \mathbb{R}^{n\times p}$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, denote the regressor and the control effectiveness, respectively. Consistent with the literature on state and parameter estimation, the control signal u and the system state x are assumed to be bounded.

The objective is to develop a real-time state observer for online state estimation of x using input u and output y alongside a parameter estimation scheme, which uses memory from recorded data to provide parameter estimates denoted as $\hat{\theta}$. The following assumption is necessary to facilitate the development and analysis of the method presented in this paper.

Assumption 1: The functions Y and g are known, their derivatives exist on the compact set $\mathcal{C} \subset \mathbb{R}^n$ containing the origin and satisfy the element-wise bounds

$$(K_{y_1})_{j,k} \leq \left(\frac{\partial(Y(x))_{j,i}}{\partial(x)_k}\right)\theta_i \leq (K_{y_2})_{j,k}, \quad (2)$$

$$(K_{g_1})_{j,k} \leq \left(\frac{\partial(g(x))_{j,l}}{\partial(x)_k}\right)u_l \leq (K_{g_2})_{j,k}, \quad (3)$$

for all $x \in \mathcal{C}$, $u \in \mathcal{U}$, $\theta \in \Theta$, $i = 1,\ldots,p$, $j,k = 1,\ldots,n$, and $l = 1,\ldots,m$, where $(\cdot)_i$, $(\cdot)_j$, $(\cdot)_{i,k}$, and $(\cdot)_{j,k}$ denote the element of the array $(\cdot)$ at the index indicated by the subscript.

Remark 1: The conditions stated in Assumption 1 are commonly required in several observer design schemes (see, e.g., [5], [7], [27], [28]).

Sufficient conditions involving multiplier matrices that characterize the affine system will be presented in the following section, along with the design of the state observer.

III. State Observer Design

This section presents the development of a state observer that generates estimates of x by employing an extended Luenberger-like observer. To facilitate the observer design, the nonlinear dynamics described in (1) is expressed as

$$\dot{x} = Ax + F_\theta(x,\theta) + G_u(x,u), \quad y = Cx, \quad (4)$$

where $A = (K_{y_1} + K_{g_1})$, $F_\theta(x,\theta) = -K_{y_1}x + Y(x)\theta$ and $G_u(x,u) = -K_{g_1}x + \sum_{i=1}^{N} g_i(x)(u)_i$. Assumption 1 implies that the derivatives of F_θ and G_u satisfy the element-wise inequalities

$$0 \leq \frac{\partial(F_\theta(x))_j}{\partial(x)_k} \leq (K_{y_2})_{j,k} - (K_{y_1})_{j,k}, \quad (5)$$

$$0 \leq \frac{\partial(G_u(x,u))_j}{\partial(x)_k} \leq (K_{g_2})_{j,k} - (K_{g_1})_{j,k}, \quad (6)$$

for all $j,k = 1,\ldots,n$. Using the derivative bounds, a state observer with three correction terms is designed as

$$\dot{\hat{x}} = A\hat{x} + F_\theta[\hat{x} + l_1(y - C\hat{x}), \hat{\theta}] + G_u[\hat{x} + l_2(y - C\hat{x}), u] + L(y - C\hat{x}), \quad (7)$$

where $\hat{x} \in \mathbb{R}^n$ is the estimate of x, $l_1 \in \mathbb{R}^{n\times q}$, $l_2 \in \mathbb{R}^{n\times q}$, and $L \in \mathbb{R}^{n\times q}$ are observer gains, $l_1(y - C\hat{x})$ and $l_2(y - C\hat{x})$ are nonlinear injection terms and $L(y - C\hat{x})$ is a linear correction term. With the state estimation error defined as $\tilde{x} := x - \hat{x}$, the estimation error dynamics is given by

$$\dot{\tilde{x}} = (A-LC)\tilde{x} + F_\theta(x,\hat{\theta}) + G_u(x,u) - F_\theta[\hat{x} + l_1(y - C\hat{x}), \hat{\theta}] - G_u[\hat{x} + l_2(y - C\hat{x}), u] + F_\theta(x,\tilde{\theta}). \quad (8)$$

where $F_\theta(x,\tilde{\theta}) := F_\theta(x,\theta) - F_\theta(x,\hat{\theta})$. Let the parameter estimation error be defined as $\tilde{\theta} := \theta - \hat{\theta}$. To facilitate the design of the state observer, the following assumption is made about the set Θ, which contains θ.

Assumption 2: There exist a known constant $\overline{\theta} \in \mathbb{R}$ such that $\|\theta\| \leq \overline{\theta}$.

Remark 2: Assumption 2 is used to implement a parameter projection algorithm that ensures $\hat{\theta}$ stays within a bounded convex set $\Theta \subset \mathbb{R}^p := \{\hat{\theta} \mid h(\hat{\theta}) \leq 0\}$, where $h(\hat{\theta}) := \hat{\theta}^\mathsf{T}\hat{\theta} - \overline{\theta}^2$ and $\nabla_{\hat{\theta}} h(\hat{\theta}) := 2\hat{\theta}$ (cf. [12, Example 4.4.2]).

Let $\mathcal{D} \subset \{\tilde{x} \in \mathbb{R}^n : x, \hat{x} \in \mathcal{C}\}$, the difference functions between the uncertain system components and their estimates can then be characterized using the difference functions $\phi_y : \mathbb{R}_{\geq 0} \times \mathcal{D} \times \Theta \to \mathbb{R}^n$ and $\phi_g : \mathbb{R}_{\geq 0} \times \mathcal{D} \to \mathbb{R}^n$, defined as $\phi_y(t,\tilde{x},\hat{\theta}) := F_\theta(x,\hat{\theta}) - F_\theta[\hat{x} + l_1(y - C\hat{x}), \hat{\theta}]$, and $\phi_g(t,\tilde{x}) := G_u(x,u) - G_u[\hat{x} + l_2(y - C\hat{x}), u]$, respectively. The observer error dynamics in (8) can then be expressed as

$$\dot{\tilde{x}} = (A - LC)\,\tilde{x} + \phi_y(t,\tilde{x},\hat{\theta}) + \phi_g(t,\tilde{x}) + F_\theta(x,\tilde{\theta}). \quad (9)$$

According to the differential mean value theorem (DMVT) [29, Theorem 2.1], provided Assumption 1 and Assumption 2 hold, the difference functions ϕ_y and ϕ_g are guaranteed to be bounded as

$$\overline{K}_{y_1}(\mathbb{I}_n - l_1C)\tilde{x} \leq \phi_y(t,\tilde{x},\hat{\theta}) \leq \overline{K}_{y_2}(\mathbb{I}_n - l_1C)\tilde{x}, \text{ and} \quad (10)$$

$$\overline{K}_{g_1}(\mathbb{I}_n - l_2C)\tilde{x} \leq \phi_g(t,\tilde{x}) \leq \overline{K}_{g_2}(\mathbb{I}_n - l_2C)\tilde{x}. \quad (11)$$

where $\overline{K}_{y_1} = 0_{n\times n}$, $\overline{K}_{y_2} = K_{y_2} - K_{y_1}$, $\overline{K}_{g_1} = 0_{n\times n}$, $\overline{K}_{g_2} = K_{g_2} - K_{g_1}$ and the notation $\mathbb{I}_n$ represents an n by n identity matrix. To establish the stability of the state estimation error dynamics, it suffices to rely on the sector information provided by the compact set $\mathcal{C}$, which is defined by the Jacobian bounds presented in (5), and (6) and constraints ϕ_y, and ϕ_g. Specifically, the inequalities in (10) and (11) can be used to obtain the following bounds

$$[\phi_y(t,\tilde{x},\hat{\theta})]^\mathsf{T}[\phi_y(t,\tilde{x},\hat{\theta}) - \overline{K}_{y_2}(\mathbb{I}_n - l_1C)\tilde{x}] \leq 0, \text{ and} \quad (12)$$

$$[\phi_g(t,\tilde{x})]^\mathsf{T}[\phi_g(t,\tilde{x}) - \overline{K}_{g_2}(\mathbb{I}_n - l_2C)\tilde{x}] \leq 0, \quad (13)$$

which can be expressed in their quadratic forms as

$$\begin{bmatrix}\tilde{x}\\ \phi_y\end{bmatrix}^\mathsf{T}\begin{bmatrix}\mathbb{I}_n - l_1C & 0\\ 0 & \mathbb{I}_n\end{bmatrix}^\mathsf{T} M_y \begin{bmatrix}\mathbb{I}_n - l_1C & 0\\ 0 & \mathbb{I}_n\end{bmatrix}\begin{bmatrix}\tilde{x}\\ \phi_y\end{bmatrix} \leq 0, \text{ and} \quad (14)$$

$$\begin{bmatrix}\tilde{x}\\ \phi_g\end{bmatrix}^\mathsf{T}\begin{bmatrix}\mathbb{I}_n - l_2C & 0\\ 0 & \mathbb{I}_n\end{bmatrix}^\mathsf{T} M_g \begin{bmatrix}\mathbb{I}_n - l_2C & 0\\ 0 & \mathbb{I}_n\end{bmatrix}\begin{bmatrix}\tilde{x}\\ \phi_g\end{bmatrix} \leq 0, \quad (15)$$

with their corresponding multiplier matrices designed as

$$M_y = \begin{bmatrix} 0 & -\frac{K_{y_2}^\top - K_{y_1}^\top}{2} \\ -\frac{K_{y_2}-K_{y_1}}{2} & \mathbb{I}_n \end{bmatrix}, \text{and} \tag{16}$$

$$M_g = \begin{bmatrix} 0 & -\frac{K_{g_2}^\top - K_{g_1}^\top}{2} \\ -\frac{K_{g_2}-K_{g_1}}{2} & \mathbb{I}_n \end{bmatrix}. \tag{17}$$

A Lyapunov-based analysis that uses the above inequalities to establish boundedness of the state estimation error for all $t \in \mathbb{R}_{\geq 0}$ is presented in Section V.

IV. Parameter Estimator Design

The parameter estimator to be designed in this section relies on the fact that the difference between the state estimates at time t and time $t - \varsigma$, where $\varsigma \in \mathbb{R}_+$ denotes the time delay, can be expressed as an affine function of the parameters θ and a residual that reduces with reducing state estimation errors as described in the following Lemma.

Lemma 1: If x, $\hat{x} \in \mathcal{C}$ and if Assumption 1 holds, for all $\varsigma \geq 0$ and for all $t \geq \varsigma$, the state estimates satisfy $\hat{x}(t) - \hat{x}(t-\varsigma) = \hat{\mathcal{Y}}(t)\theta + \hat{\mathcal{G}}_u(t) + \mathcal{E}(t)$, where $\hat{\mathcal{Y}}(t) := \int_{t-\varsigma}^{t} Y(\hat{x}(\tau))d\tau$, $\hat{\mathcal{G}}_u(t) := \int_{t-\varsigma}^{t} g(\hat{x}(\tau))u(\tau)d\tau$, and $\mathcal{E}(t) = O\left(\sup_{\sigma\in[t-\varsigma,t]} \|\tilde{x}(\sigma)\|\right)$.

Proof: Integrating the dynamics in (1) yields

$$x(t) - x(t-\varsigma) = \int_{t-\varsigma}^{t} Y(x(\tau))\theta + g(x(\tau))u(\tau)d\tau \tag{18}$$

By adding and subtracting $x(t)$ and $x(t-\varsigma)$, the difference $\hat{x}(t) - \hat{x}(t-\varsigma)$ can be expressed as

$$\hat{x}(t) - \hat{x}(t-\varsigma) = -\tilde{x}(t) + \tilde{x}(t-\varsigma) + x(t) - x(t-\varsigma). \tag{19}$$

Substituting from (18), adding and subtracting the integral $\int_{t-\varsigma}^{t} Y(\hat{x}(\tau))\theta + g(\hat{x}(\tau))u(\tau)d\tau$, and simplifying yields

$$\begin{aligned}\hat{x}(t)-\hat{x}(t-\varsigma) &= \int_{t-\varsigma}^{t} Y(\hat{x}(\tau))\theta d\tau + \int_{t-\varsigma}^{t} g(\hat{x}(\tau))u(\tau)d\tau \\ &- \tilde{x}(t) + \tilde{x}(t-\varsigma) + \int_{t-\varsigma}^{t} \tilde{Y}(x(\tau),\hat{x}(\tau))\theta d\tau \\ &+ \int_{t-\varsigma}^{t} \tilde{g}(x(\tau),\hat{x}(\tau))u(\tau)d\tau \end{aligned} \tag{20}$$

where $\tilde{Y}(x(\tau),\hat{x}(\tau)) := Y(x(\tau)) - Y(\hat{x}(\tau))$ and $\tilde{g}(x(\tau),\hat{x}(\tau)) := g(x(\tau)) - g(\hat{x}(\tau))$. If $x, \hat{x} \in \mathcal{C}$ and Assumption 1 holds, then the DMVT can be invoked to obtain $\hat{x}(t) - \hat{x}(t-\varsigma) = \hat{\mathcal{Y}}(t)\theta + \hat{\mathcal{G}}_u(t) + \mathcal{E}(t)$ where the residual term satisfies $\mathcal{E}(t) = O\left(\sup_{\sigma\in[t-\varsigma,t]} \|\tilde{x}(\sigma)\|\right)$. ■

Lemma 1 implies that the parameter estimation error at any time t can be expressed as $\hat{\mathcal{Y}}(t)\tilde{\theta}(t) = \hat{x}(t) - \hat{x}(t-\varsigma) - \hat{\mathcal{G}}_u(t) - \hat{\mathcal{Y}}(t)\hat{\theta}(t) - \mathcal{E}(t)$, which motivates the update law

$$\dot{\hat{\theta}} = \begin{cases} k_\theta \Gamma \phi, & \text{if } \hat{\theta}^\top\hat{\theta} < \overline{\theta}^2 \text{or if} \\ & \hat{\theta}^\top\hat{\theta} = \overline{\theta}^2 \text{and } (k_\theta\Gamma\phi)^\top\hat{\theta} \leq 0 \\ \left(\mathbb{I}_p - \frac{\Gamma\hat{\theta}\hat{\theta}^\top}{\hat{\theta}^\top\Gamma\hat{\theta}}\right)k_\theta\Gamma\phi, & \text{otherwise} \end{cases} \tag{21}$$

where $\phi(t) := \sum_{i=1}^{N}\left(\frac{\hat{\mathcal{Y}}(t_i)}{1+\kappa\|\hat{\mathcal{Y}}(t_i)\|^2}\right)^\top \left(\hat{x}(t_i) - \hat{x}(t_i-\varsigma) - \hat{\mathcal{G}}_u(t_i) - \hat{\mathcal{Y}}(t_i)\hat{\theta}\right)$, $\kappa \in \mathbb{R}_{>0}$ is the normalization gain, and $k_\theta \in \mathbb{R}_{>0}$ is the CL gain. The matrix $\Gamma \in \mathbb{R}^{p\times p}$ is the least-squares gain matrix updated as

$$\dot{\Gamma} = \begin{cases} \beta_1\Gamma - k_\theta\Gamma\frac{\hat{\mathcal{Y}}(t_i)^\top\hat{\mathcal{Y}}(t_i)}{1+\kappa\|\hat{\mathcal{Y}}(t_i)\|^2}\Gamma, & \text{if } \hat{\theta}^\top\hat{\theta} < \overline{\theta}^2 \text{or if} \\ & \hat{\theta}^\top\hat{\theta} = \overline{\theta}^2 \text{and } (k_\theta\Gamma\phi)^\top\hat{\theta} \leq 0 \\ 0 & \text{otherwise} \end{cases} \tag{22}$$

where $\beta_1 \in \mathbb{R}_{>0}$ is a constant adaptation gain. The update law relies on the time delay ς, and a history stack $\mathcal{H}$. The history stack represents a set of piecewise constant functions that can be expressed as

$$\hat{\mathscr{X}} := \begin{bmatrix} \hat{x}(t_1) - \hat{x}(t_1-\varsigma) \\ \vdots \\ \hat{x}(t_N) - \hat{x}(t_N-\varsigma) \end{bmatrix}, \hat{\mathscr{Y}} := \begin{bmatrix} \hat{\mathcal{Y}}(t_1) \\ \vdots \\ \hat{\mathcal{Y}}(t_N) \end{bmatrix}, \hat{\mathscr{G}}_u := \begin{bmatrix} \hat{\mathcal{G}}_u(t_1) \\ \vdots \\ \hat{\mathcal{G}}_u(t_N) \end{bmatrix} \tag{23}$$

where $\hat{\mathscr{X}} \in \mathbb{R}^{nN}$, $\hat{\mathscr{Y}} \in \mathbb{R}^{nN\times p}$ and $\hat{\mathscr{G}}_u \in \mathbb{R}^{nN}$. The integral terms in the adaptive update law can be calculated as $\hat{\mathcal{Y}}(t) = \hat{I}_Y(t) - \hat{I}_Y(t-\varsigma)$ and $\mathcal{G}_u(t) = \hat{I}_{g_u}(t) - \hat{I}_{g_u}(t-\varsigma)$, where $\hat{I}_Y(t) = \int_0^t Y(\hat{x}(\tau))d\tau$ and $\hat{I}_{g_u}(t) = \int_0^t g(\hat{x}(\tau))u(\tau)d\tau$, are computed by solving

$$\dot{\hat{I}}_Y = Y(\hat{x}) \text{ and } \dot{\hat{I}}_{g_u} = g(\hat{x})u \tag{24}$$

starting from the initial conditions $I_{Y,0} = 0_{n\times p}$ and $I_{g_u,0} = 0_{n\times 1}$. Using Lemma 1, the parameter estimation error dynamics can be expressed as

$$\dot{\tilde{\theta}} = -k_\theta\Gamma\sum_{i=1}^{N}\frac{\hat{\mathcal{Y}}(t_i)^\top\hat{\mathcal{Y}}(t_i)}{1+\kappa\|\hat{\mathcal{Y}}(t_i)\|^2}\tilde{\theta} - k_\theta\Gamma\sum_{i=1}^{N}\frac{\hat{\mathcal{Y}}(t_i)^\top\mathcal{E}(t_i)}{1+\kappa\|\hat{\mathcal{Y}}(t_i)\|^2}. \tag{25}$$

It is clear from (25) that for the parameter estimation error to be bounded, the matrix $\sum_{i=1}^{N}\frac{\hat{\mathcal{Y}}(t_i)^\top\hat{\mathcal{Y}}(t_i)}{1+\kappa\|\hat{\mathcal{Y}}(t_i)\|^2}$ needs to be positive definite, which can be ensured if the trajectories are sufficiently informative and the data $(\hat{x}(t_i) - \hat{x}(t-\varsigma), \hat{\mathcal{Y}}(t_i), \hat{\mathcal{G}}_{u_i})_{i=1}^{N}$ stored in the history stack $\mathcal{H}$ are recorded carefully. The following assumption formalizes this requirement.

Assumption 3: For a given $N \in \mathbb{N}$, there exist a set of time instances $\{t_i\}_{i=1}^{N}$ such that $\lambda_{\min}\left(\sum_{i=1}^{N}\frac{\hat{\mathcal{Y}}(t_i)^\top\hat{\mathcal{Y}}(t_i)}{1+\kappa\|\hat{\mathcal{Y}}(t_i)\|^2}\right) = \underline{c} > 0$.

In the following, a history stack that meets the eigenvalue condition in Assumption 3 is called *full rank*.

Since the convergence rate of the parameter estimation errors depends on the lower bound $\underline{c}$ on the minimum eigenvalue, a minimum eigenvalue maximization algorithm is utilized for the selection of the time instances $\{t_i\}_{i=1}^{N}$ (see, for example, [3]). The algorithm presented in Algorithm 1 replaces an existing data point $\left(\hat{x}_i - \hat{x}_{i-\varsigma}, \hat{\mathcal{Y}}_i, \hat{\mathcal{G}}_{u_i}\right)$, with a new data point $\left(\hat{x}^* - \hat{x}^{*-}, \hat{\mathcal{Y}}^*, \hat{\mathcal{G}}_u^*\right)$, for some $i \in 1,\ldots,N$, where $\hat{x}^* - \hat{x}^{*-} := \hat{x}(t) - \hat{x}(t-\varsigma)$, $\hat{\mathcal{Y}}^* := \hat{\mathcal{Y}}(t)$ and $\hat{\mathcal{G}}_u^* := \hat{\mathscr{G}}_u(t)$, only if the condition

$$\lambda_{\min}\left(\sum_{i\neq j}\sigma_i\hat{\mathcal{Y}}_i^\top\hat{\mathcal{Y}}_i + \sigma_j\hat{\mathcal{Y}}_j^\top\hat{\mathcal{Y}}_j\right)$$

$$< \frac{\lambda_{\min}\left(\sum_{i\neq j}\sigma_i\hat{\mathcal{Y}}_i^{\top}\hat{\mathcal{Y}}_i + \sigma^*\hat{\mathcal{Y}}^{\top}\hat{\mathcal{Y}}^*\right)}{(1+\delta)} \tag{26}$$

holds. Here, $\lambda_{\min}(\cdot)$ denotes the minimum eigenvalue of a matrix, δ is a constant that can be adjusted, $\sigma_i := \frac{1}{1+\kappa\|\hat{\mathcal{Y}}_i\|^2}$, $\sigma_j := \frac{1}{1+\kappa\|\hat{\mathcal{Y}}_j\|^2}$, and $\sigma^* := \frac{1}{1+\kappa\|\hat{\mathcal{Y}}^*\|^2}$. The availability of accurate state estimates is required for precise parameter estimation. However, the initial history stack, recorded during transients, may contain inaccurate data, requiring a purge of the history stack once more accurate state estimates become available. In such cases, newer state estimates are preferred, subject to the conditions of Theorem 1. A greedy purging algorithm based on dwell time is employed to ensure estimator stability while utilizing newer data. This algorithm uses two history stacks: a main stack denoted as $\mathcal{H}$ and a transient stack labeled $\mathcal{G}$. The transient stack is filled until a sufficient dwell time $\mathcal{T}$ has elapsed. Then, the main stack is purged, and the transient stack is copied into the main stack. This approach enables the use of newer, more accurate data while maintaining estimator stability.

V. Stability Analysis

In this section, stability analysis of the joint state and parameter estimation architecture will be carried out using Lyapunov methods.

A. Analysis of state observer

The following Theorem establishes local uniform ultimate boundedness of the state estimation errors.

Theorem 1: Provided Assumption 1 and 2 hold, there exists a constant symmetric positive definite matrix, P, and three observer gains, l_1, l_2 and L, that satisfy the matrix inequality,

$$\begin{bmatrix} \begin{pmatrix}(A-LC)^{\top}P\\ +P(A-LC)\end{pmatrix}+2\alpha P & P-(I-l_1C)^{\top}(M_y)_{22} & P-(I-l_2C)^{\top}(M_g)_{22} \\ P-(M_y)_{21}(I-l_1C) & -(M_y)_{22} & 0 \\ P-(M_g)_{21}(I-l_2C) & 0 & -(M_g)_{22}\end{bmatrix} < 0, \tag{27}$$

then observer error in (8) is locally uniformly ultimately bounded.

Proof: Consider the continuously differentiable candidate Lyapunov function, $W : \mathcal{D} \to \mathbb{R}$ defined as

$$W(\tilde{x}) := \tilde{x}^{\top}P\tilde{x}, \tag{28}$$

which satisfies $\lambda_{\min}(P)\|\tilde{x}\|^2 \leq W(\tilde{x}) \leq \lambda_{\max}(P)\|\tilde{x}\|^2$. Since P is a constant symmetric positive definite matrix, both eigenvalues are positive. On the set, $\mathcal{D}$, the orbital derivative of the Lyapunov function along the trajectories of (8) can be expressed as

$$\dot{W}(\tilde{x},t) := \begin{bmatrix}\tilde{x}\\ \phi_f\\ \phi_g\end{bmatrix}^{\top}\begin{bmatrix}(A-LC)^{\top}P+P(A-LC) & P & P\\ P & 0 & 0\\ P & 0 & 0\end{bmatrix}\begin{bmatrix}\tilde{x}\\ \phi_f\\ \phi_g\end{bmatrix} + \tilde{x}^{\top}PF_{\theta}(x,\tilde{\theta}) + F_{\theta}(x,\tilde{\theta})P\tilde{x}. \tag{29}$$

Provided the matrix inequalities in (27) is satisfied for some constant $\alpha \in \mathbb{R}_+$, the multiplier matrices and sector

Algorithm 1 Algorithm for Adaptive History Stack Observer. At each time instance t, τ_1 stores the last time an event occurred, τ_2 stores the last time instance $\mathcal{H}$ was purged, λ stores the highest minimum eigenvalue encountered so far, $\mathcal{T}$ denotes the dwell time, λ^* denotes some user selected eigenvalue threshold, t^* denotes some user selected sampling rate and $\xi \in (0, 1]$ is a threshold for purging.

Require: $t_f \in \mathbb{R}_{\geq t_0}$, $t^* \in \mathbb{R}_+$, $T \in \mathbb{R}_{\geq 0}$, $\lambda^* \geq 0$
$\hat{\mathscr{X}} \leftarrow 0$, $\hat{\mathscr{Y}} \leftarrow 0$, $\hat{\mathscr{G}}_u \leftarrow 0$, $\tau_1 \leftarrow 0$, $\tau_2 \leftarrow 0$ ▷ Global variables
$\lambda \leftarrow \min(\mathrm{eig}(\hat{\mathscr{Y}}^{\top}\hat{\mathscr{Y}}))$, $t_0 \leftarrow 0$, $\hat{x}_0 \leftarrow \hat{x}(t_0)$, $\hat{\theta}_0 = \hat{\theta}(t_0)$
while $t_0 < t_f$ **do**
  integrate DDEs in (21), (22) and (24) over $[t_0, t_f]$
  if $(t - \tau_1) \geq t^*$ **then**
    if $t \geq \varsigma$ **then**
      stop integration, an event has occurred
      $j \leftarrow \mathrm{argmax}_{i=1:N}\left\{\min\left\{\mathrm{eig}\left(\hat{\mathscr{Y}}^{\top}\hat{\mathscr{Y}} - \hat{\mathcal{Y}}_i^{\top}\hat{\mathcal{Y}}_i + \hat{\mathcal{Y}}^{\top}\hat{\mathcal{Y}}\right)\right\}\right\}$
      if $\max_{i=1:N}\left\{\min\left\{\mathrm{eig}\left(\hat{\mathscr{Y}}^{\top}\hat{\mathscr{Y}} - \hat{\mathcal{Y}}_i^{\top}\hat{\mathcal{Y}}_i + \hat{\mathcal{Y}}^{\top}\hat{\mathcal{Y}}\right)\right\}\right\} - \lambda \geq \lambda^*$ **then**
        $\lambda \leftarrow \max_{i=1:N}\left\{\min\left\{\mathrm{eig}\left(\hat{\mathscr{Y}}^{\top}\hat{\mathscr{Y}} - \hat{\mathcal{Y}}_i^{\top}\hat{\mathcal{Y}}_i + \hat{\mathcal{Y}}^{\top}\hat{\mathcal{Y}}\right)\right\}\right\}$
        $\{\hat{\mathscr{Y}}_i\}_{i=n(j-1)}^{nj} \leftarrow \hat{\mathcal{Y}}(t)$
        $\{\hat{\mathscr{G}}_{u_i}\}_{i=n(j-1)}^{nj} \leftarrow \hat{\mathcal{G}}_u(t)$
        $\{\hat{\mathscr{X}}_i\}_{i=n(j-1)}^{nj} \leftarrow \hat{x}(t) - \hat{x}(t-\varsigma)$
        if $\mathcal{G}$ is not full **then**
          add the data points to $\mathcal{G}$
        else
          add the data points to $\mathcal{G}$ if (26) holds
        end if
        if $\min(\mathrm{eig}(\hat{\mathscr{Y}}^{\top}\hat{\mathscr{Y}})) \geq \xi\lambda$ **then**
          if $(t - \tau_2) \geq \mathcal{T}(t)$ **then**
            $\mathcal{H} \leftarrow \mathcal{G}$, $\mathcal{G} \leftarrow 0$, and $\tau_2 \leftarrow t$
            if $\lambda < \min(\mathrm{eig}(\hat{\mathscr{Y}}^{\top}\hat{\mathscr{Y}}))$ **then**
              $\lambda \leftarrow \min(\mathrm{eig}(\hat{\mathscr{Y}}^{\top}\hat{\mathscr{Y}}))$
            end if
          end if
          $t_0 \leftarrow t$, $x_0 \leftarrow x(t)$, $\hat{\theta}_0 \leftarrow \hat{\theta}(t)$
          $I_{Y,0} \leftarrow \hat{I}_Y(t)$, $\hat{I}_{g_u,0} \leftarrow \hat{I}_{g_u}(t)$
        end if
      end if
    else
      no event, keep on integrating the DDEs
    end if
    $\tau_1 \leftarrow t$ ▷ Set this even if a new event is not detected
  end if
  no event, keep on integrating the DDEs
end while

conditions formulated in (14) and (15), the S-Procedure Lemma [30], Assumption 1, and Assumption 2 can be used to guarantee that the orbital derivative of W is bounded as (cf. [9], [31])

$$\dot{W}(\tilde{x},t) \leq -2\alpha W(\tilde{x}) + 2\lambda_{\max}(P)\overline{F}_{\tilde{\theta}}\|\tilde{x}\|, \tag{30}$$

for all $\tilde{x} \in \mathcal{D}$, where $\max_{x\in\mathcal{C}}\|F_{\theta}(x,\tilde{\theta})\| \leq \overline{F}_{\tilde{\theta}}$ for some $\overline{F}_{\tilde{\theta}} \in \mathbb{R}_+$. Hence, the orbital derivative is bounded along the trajectories of (9) as

$$\dot{W}(\tilde{x},t) \leq -\alpha W(\tilde{x}), \forall\tilde{x} \in \mathcal{D}, \|\tilde{x}\| \geq \xi > 0. \tag{31}$$

where $\xi = \frac{2\lambda_{\max}(P)\overline{F}_{\tilde{\theta}}}{\alpha\lambda_{\min}(P)}$. Invoking [32, Thereom 4.18], the state estimation error is locally uniformly ultimately bounded. And the ultimate bound on $\tilde{x}$ can be estimated as $\limsup_{t\to\infty}\|\tilde{x}\| := \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\xi$.

Remark 3: The observer design is only valid if the control input remains bounded and the system trajectories remain within the compact set $\mathcal{C}$ where the bounds on the Jacobians in (5), and (6), respectively, are valid.

Remark 4: The matrix inequality in (27) can be reformulated as a linear matrix inequality (LMI) using the typical variable substitution method. Indeed, substituting $L = P^{-1}R$ in (27), the matrix P and the observer gains L, l_1 and l_2 can be obtained by solving the LMI

$$\begin{bmatrix} \begin{pmatrix} A^{\top}P+PA \\ -C^{\top}R^{\top}-RC \end{pmatrix}+2\alpha P & P-(I-l_1C)^{\top}(M_y)_{22} & P-(I-l_2C)^{\top}(M_g)_{22} \\ P-(M_y)_{21}(I-l_1C) & -(M_y)_{22} & 0 \\ P-(M_g)_{21}(I-l_2C) & 0 & -(M_g)_{22} \end{bmatrix} < 0, \tag{32}$$

for P, R, $l_1$and l_2.

■

B. Analysis of Parameter Estimator

In order to rigorously analyze the convergence properties of the parameter estimation error, a precise definition of "finitely informative" and "persistently informative" data in the history stack is presented below.

Definition 1: [33] The signal $(\hat{x}, u)$ is called finitely informative (FI) if there exist time instances $0 \le t_1 < t_2 < \ldots < t_N$, for some finite positive integer N, such that the resulting history stack is full rank and persistently informative (PI) if, for any $T \ge 0$, there exist time instances $T \le t_1 < t_2 < \ldots < t_N$ such that the resulting history stack is full rank.

The subsequent theorem establishes that the parameter estimation error $\tilde{\theta}$ converges to a neighborhood of the origin if Assumption 3 holds and the data are sufficiently informative, as per Definition 1. To facilitate the analysis, given s in $\mathbb{N}$, let $\mathcal{H}_s$ denote the history stack that is active during the time interval $I_s := \{t \mid \rho(t) = s\}$ containing the data $\left\{(\hat{\mathscr{X}}_{si}, \hat{\mathscr{Y}}_{si}, \hat{\mathscr{G}}_{fu_{si}})\right\}_{i=1,\ldots,N}$, where $\rho : \mathbb{R}_{\ge 0} \to \mathbb{N}$ denotes a switching signal that satisfies initial condition $\rho(0) = 1$ and for any time t in the domain of the signal, $\rho(t) = j+1$, where j denotes the number of times the update $\mathcal{H} \leftarrow \mathcal{G}$ has been carried out over the time interval 0 to t. To also facilitate the analysis and simplify notation, let $\varPsi_s : \mathbb{R}_{\ge 0} \to \mathbb{R}^{p\times p}$ and $Q_s : \mathbb{R}_{\ge 0} \to \mathbb{R}^{n\times p}$ be defined as $\varPsi_s := \sum_{i=1}^{N} \frac{\hat{\mathcal{Y}}_{si}^{\top}\hat{\mathcal{Y}}_{si}}{1+\kappa\|\hat{\mathcal{Y}}_{si}\|^2}$ and $Q_s := \sum_{i=1}^{N} \frac{\hat{\mathcal{Y}}_{si}^{\top}\mathcal{E}_{si}}{1+\kappa\|\hat{\mathcal{Y}}_{si}\|^2}$. Using this notation, the dynamics of the parameter estimation error in (25) and (22) can be expressed as

$$\dot{\tilde{\theta}} = -k_\theta\Gamma\varPsi_s\tilde{\theta} - k_\theta\Gamma Q_s \text{ and } \dot{\Gamma} = \beta_1\Gamma - k_\theta\Gamma\varPsi_s\Gamma \tag{33}$$

respectively. It is important to note that the functions $\varPsi_s$ and Q_s are piece-wise continuous. Thus, the trajectories of (33) are defined in the sense of Carathéodory [3], [34]. Using arguments similar to [1, Theorem 1], provided the conditions of Theorem 1 are satisfied, and the states and state estimation errors remain within the compact sets $\mathcal{C}$ and $\mathcal{D}$, respectively, over the time interval I_{s-1} in which the history stack was recorded, then using the error bound developed in Lemma 1 the error terms can be bounded as

$$\|\mathcal{E}_{si}\| \le L_e\overline{e}_s, \forall i \in \{1,\ldots,N\}, \forall \tilde{x} \in \mathcal{D}, \tag{34}$$

where $\overline{e}_s := \sup_{t\in I_{s-1}}\|\tilde{x}(t)\|$ and $L_e \in \mathbb{R}_+$ is a constant.

Theorem 2: If the state and parameters of the system in (1) are estimated using state and parameter estimators that satisfy the conditions of Theorem 1 and Assumption 3, the signal $(\hat{x}, u)$ is FI, $\mathcal{H}$ is populated using Algorithm 1, and if the excitation lasts long enough for two purging events (i.e. $\mathcal{H}_3$ is full rank), then the trajectories of the parameter estimation error are ultimately bounded.

Proof: Consider the candidate Lyapunov function $V : \Theta \times \mathbb{R}_{\ge 0} \to \mathbb{R}$ defined as,

$$V(\tilde{\theta}, t) := \frac{1}{2}\tilde{\theta}^{\top}\Gamma^{-1}(t)\tilde{\theta}. \tag{35}$$

Using arguments similar to those presented in [12, Section 4.4.2], provided (3) holds and $\lambda_{\min}\{\Gamma(0)^{-1}\} > 0$, the update law in (22) ensures that the least squares update law satisfies

$$\underline{\Gamma}\mathbb{I}_p \le \Gamma(t) \le \overline{\Gamma}\mathbb{I}_p, \forall t \in \mathbb{R}_{\ge 0} \tag{36}$$

for some $\overline{\Gamma}, \underline{\Gamma} \in \mathbb{R}_+$, where $\mathbb{I}_p$ denotes a $p \times p$ identity matrix. Applying the bound in (36), the candidate Lyapunov function satisfies the following inequality

$$\frac{1}{2\overline{\Gamma}}\|\tilde{\theta}\|^2 \le V(\tilde{\theta}, t) \le \frac{1}{2\underline{\Gamma}}\|\tilde{\theta}\|^2, \forall t \in \mathbb{R}_{\ge 0}. \tag{37}$$

Using arguments similar to those presented in [12, Theorem 4.4.1], the orbital derivative of V can be bounded as, $\dot{V}_s(\tilde{\theta}, t) \le -\frac{1}{2}\underline{a}\|\tilde{\theta}\|^2 + k_\theta\overline{Q}_s\|\tilde{\theta}\|$, where $\underline{a} := k_\theta\underline{c} + \frac{\beta_1}{\overline{\Gamma}}$, $\underline{c}$ is defined in Assumption 3 and $\overline{Q}_s$ is a positive constant such that $\overline{Q}_s \ge \|Q_s\|$. Using the completion of squares, the orbital derivative is then bounded for all $t \in \mathbb{R}_{\ge 0}$ as

$$\dot{V}_s(\tilde{\theta}, t) \le -\frac{1}{4}\underline{a}\|\tilde{\theta}\|^2, \forall\|\tilde{\theta}\| \ge \varrho(\|\mu\|) \tag{38}$$

where $\varrho(\|\mu\|) := \left(\frac{4k_\theta}{\underline{a}}\right)\|\mu\|^2$ and $\mu := \sqrt{\overline{Q}_s}$. Hence, the conditions of [32, Theorem 4.19] are satisfied, and it can be concluded that (33) is input-to-state stable with state $\tilde{\theta}$ and input μ. If Algorithm 1 is implemented and if the signal $(\hat{x}, u)$ is FI, then there exists a time instance T_s, such that for all $t \ge T_s$, the history stack remains unchanged. And as a result, using [32, Exercise 4.58], an ultimate bound on $\tilde{\theta}$ can be estimated as

$$\limsup_{t\to\infty}\|\tilde{\theta}(t)\| \le \overline{\theta}(T_s) := \sqrt{\frac{\overline{\Gamma}}{\underline{\Gamma}}}\left(\frac{4k_\theta\overline{Q}(T_s)}{\underline{a}}\right). \tag{39}$$

where $\overline{Q}(T_s)$ denotes a bound on residual error term Q_s, in the history stack $\mathcal{H}$ for all $t \ge T_s$. The parameter estimation error can be reduced by reducing the estimation errors corresponding to the state estimates stored in the history stack, which reduces Q_s. The projection algorithm and Theorem 1 imply the boundedness of all signals in the closed loop for all t. Furthermore, Theorem 1 implies that given any $\varepsilon \in \mathbb{R}_+$, the gain α can be selected large enough to ensure that $\tilde{x}$ has reached the ultimate bound before $t = T_1$ and that the ultimate bound is smaller than ε so that $\overline{e}_2 \leq \varepsilon$. Since the history stack $\mathcal{H}_3$, which is active over the interval I_3, is recorded during the interval I_2, the bounds in (34) can be used to show $\overline{Q}_3 = \frac{NL_e\overline{e}_2}{2\sqrt{\kappa}} \leq \frac{NL_e\varepsilon}{2\sqrt{\kappa}}$. As such, if $(\hat{x}, u)$ is FI with the excitation lasting long enough so that $\mathcal{H}_3$ is full rank, then (39) implies that $\limsup_{t\to\infty} \|\tilde{\theta}(t)\| \leq \sqrt{\frac{\overline{\Gamma}}{\kappa\underline{\Gamma}}}\left(\frac{2k_\theta NL_e}{\underline{a}}\right)\varepsilon$.

■

VI. Simulation

A two-state dynamical system is simulated to demonstrate the developed method's performance. Consider the dynamical system of the form in (1) with states $x = [(x)_1; (x)_2]$ where

$$Y(x) = \begin{bmatrix} (x)_2 & 0 & 0 & 0 \\ 0 & (x)_1 & (x)_2 & x_2(\cos(2(x)_1) + 2)^2 \end{bmatrix}, \quad (40)$$

$\theta = [(\theta)_1; (\theta)_2; (\theta)_3; (\theta)_4]$, $g(x) = [0; \cos(2(x)_1) + 2]$ and $C = [1; 0]^\intercal$. To satisfy Assumption 3, a controller that results in a uniformly bounded system response is needed, which is chosen to be a proportional-derivative (PD), defined as $u = -k_p((x)_1 - x_d) - k_d((x)_2 - \dot{x}_d)$, where k_p and k_d are control gains. The objective of this controller is to make the system track the trajectory $(x_d)_1(t) = (x_d)_2(t) = -\frac{1}{3}\cos(3t) - \frac{1}{2}\cos(2t)$. The initial conditions of the systems are selected as $x(0) = [2; 2]$, $\hat{x}(0) = [2.5; 1.5]$, $\hat{\theta}(0) = [0; 0; 0; 0]$ and the actual values of the unknown parameters in the system model are $(\theta)_1 = 1$, $(\theta)_2 = -1$, $\theta_3 = -0.5$, $\theta_4 = 0.5$. In order to satisfy the stability conditions

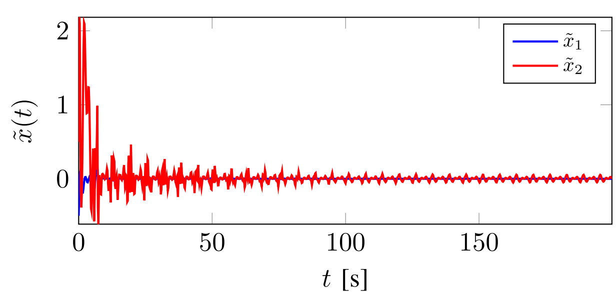

Fig. 1: Trajectory of state estimation error

of Theorem 1, the LMI in (27) is solved using SEDUMI in YALMIP on MATLAB. The objective is to obtain the three

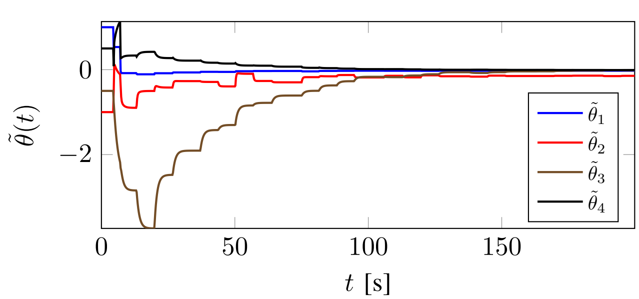

Fig. 2: Trajectory of parameter estimation error

observer gains, L, l_1, and l_2, and the symmetric positive definite matrix, P, which satisfies the LMI. The learning rate used in the LMI is $\alpha = 2$. Data is added to the history stack $\mathcal{H}$ using the minimum eigenvalue maximization algorithm detailed in Algorithm 1 with initial values given as $I_{Y,0} = 0_{2\times 4}$ and $I_{gu,0} = 0_{2\times 1}$, $\varsigma = 2$, $t^* = 0.1$, $\lambda^* = 0$. The learning gains are selected through trial and error, as $N = 25$, $k_\theta = 50$, $\beta_1 = 0.5$, $\Gamma(0) = \text{diag}([1, 1, 1, 1])$, $k_p = [50; 50]^\intercal$.

A. Results and Discussion

Figure 1 and figure 2 demonstrate that the developed state and parameter estimators are effective in driving the trajectories of state estimation errors and parameter estimation errors to the origin, respectively. This result demonstrates the effectiveness of the developed method and validates the theoretical results in Section V. The value of the constant symmetric positive definite matrix P is given as $P = [2.3886, -0.1840; -0.1840, 0.0270]$, and the value of the observer gain was found to be $L = [10.0671; 103.167]$.

VII. Conclusion

An online joint state and parameter estimation scheme for nonlinear systems using a multiplier matrix observer design and an event-based implementation of concurrent learning adaptive update laws is developed. Convergence properties of the developed method are analyzed using Lyapunov methods and validated through simulation, demonstrating local uniformly ultimately boundedness of the state estimation errors and input-to-state stability of parameter estimation errors under a finite informativity condition.

To avoid the need to compute exact Jacobian bounds and allow for relaxed LMI conditions, future work will involve developing a methodology for simultaneous state and parameter estimation via exact Takagi-Sugeno tensor-product models or polynomial rewriting of the error system, as formulated in [10], [11].

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