An Output Feedback Approach to Differential Graphical Games in Linear Multiagent Systems

Tochukwu E. Ogri[1] Muzaffar Qureshi[1] Zachary I. Bell[2] Wanjiku A. Makumi[2] Rushikesh Kamalapurkar[1]

***Abstract*— This letter presents an output feedback approach to distributed optimal formation control of linear time-invariant multiagent systems. The formation control problem is formulated as a differential graphical game problem. It is assumed that each agent receives partial error-states of its immediate neighbors. To account for the dependence of the value function of each agent on the error-states of its extended neighbors, a robust observer that estimates the error-states of the extended neighbors using partial error-states of the immediate neighbors is designed. The observer is integrated with a controller to approximate a global feedback Nash equilibrium (FNE) solution of the differential graphical game. Stability of the closed-loop system and convergence of the estimated value functions to the approximate FNE solution are established using a Lyapunov-based analysis. Simulations demonstrate the efficacy of the developed approach.**

***Index Terms*— Multiagent systems, differential graphical games, formation control, optimal control.**

I. Introduction

DISTRIBUTED control [1]–[4] is effective in controlling multiagent systems, particularly when agents in the network have limited computational, sensing, and communication capabilities. Distributed control involves designing a network of local controllers with collaborative or competitive objectives, effectively distributing the computational load across the network and mitigating the risk of system-wide failures [2]. The focus of this letter is optimal formation tracking, where individual agents track a mobile leader while maintaining a desired formation.

The multiagent optimal formation tracking problem is a multi-objective optimization problem. Several notions of optimality can be used to address multi-objective optimization problems, including game-theoretic notions such as Pareto efficiency and Nash equilibria. Game theory, often associated with competition, also provides a natural framework for defining optimality of interactions between agents in a cooperative setting [2], [3], [5]. The game-theoretic approach utilized in this paper is motivated by various centralized [6], [7] and decentralized [3], [8]–[10] techniques in the literature that study multiagent formation-tracking through the lens of feedback Nash equilibrium (FNE) solutions.

The key challenge in distributed control is that each agent in the network only has partial knowledge of the state of the multiagent system. In addition, the state of the leader is generally not directly available to follower agents in the network. In this letter, we use an observer-based approach to overcome this challenge where each agent reconstructs the state of the system using local measurements and information gathered from its direct neighbors.

In [8], decentralization is achieved under the strict assumption that the value function only depends on the local neighborhood tracking error. In [3], the authors develop a distributed differential games framework by solving a series of fictitious local differential games that capture each agent's limited knowledge of the network. The solutions of the local games are then combined to ensure stability, but not near-optimality of the entire network. In contrast, the technique developed in this paper generates approximate FNE solutions of the differential graphical game. In [10], the authors utilize spatially-exponentially decaying (SED) (A, B, Q, R) matrices to show that the optimal LQR gain K is "quasi"-SED, i.e., the influence of a neighbor on the value function of an agent diminishes exponentially with distance. Unlike [10], this letter does not impose the SED requirement on the system matrices to obtain decentralized approximate FNE policies.

In [9], the authors develop conditions on the cost functions of each agent that guarantee that the resulting FNE policies can be executed in a decentralized manner. However, knowledge of the dynamics and the cost functions of the extended neighborhood is needed to select the cost functions. Furthermore, the approach in [9] requires each agent to measure the full state of its neighbors. The observer-based approach developed in this letter utilizes only partial state feedback and does not require modification of the local cost functions to achieve decentralization.

In multi-agent differential graphical games, the FNE value functions and policies are inherently centralized, i.e., they typically depend not only on its own state but also on the states of its extended neighborhood. The key contribution of this letter is to show that distributed observers [11]–[17] designed to estimate the state of the extended neighborhood can be utilized to decentralize the execution of the FNE policies. The fact that the extended neighborhood is different for each agent necessitates a careful re-design of the information exchange component of the distributed observer. These distributed observers are then integrated with a dynamic programming-based

This research was supported in part by the Air Force Research Laboratory under contract numbers FA8651-24-1-0019 and FA8651-23-1-0006 and the Office of Naval Research under contract number N00014-21-1-2481. Any opinions, findings, or recommendations in this article are those of the author(s), and do not necessarily reflect the views of the sponsoring agencies.

[1]Department of Mechanical and Aerospace Engineering, University of Florida, Gainesville, FL, USA, email: {tochukwu.ogri, muzaffar.qureshi, rkamalapurkar} @ufl.edu.

[2]Air Force Research Laboratory, FL, USA, email: {zachary.bell.10, wanjiku.makumi} @us.af.mil.

control architecture to develop approximate FNE policies that rely only on local neighbor information.

II. Problem Formulation

A. Graph Theory

Consider a multiagent system composed of N agents described by a directed graph $\mathcal{G}(\mathcal{N}, \mathcal{E})$, where the set of nodes $\mathcal{N} = \{1, \ldots, N\}$ and the set of edges $\mathcal{E} \subset \mathcal{N} \times \mathcal{N}$ represent the agents and the communication flow between the agents, respectively. The adjacency matrix $\mathcal{A} \in \mathbb{R}^{N\times N}$ is defined as $\mathcal{A} = [a_{ij} |\, i, j \in \mathcal{N}]$, where $a_{ij} > 0$ if $(j, i) \in \mathcal{E}$ and $a_{ij} = 0$ if $(j, i) \notin \mathcal{E}$. The digraph is assumed to have no repeated edges, i.e., $(i, i) \notin \mathcal{E}, \forall i$, which implies $a_{ii} = 0, \forall i$. The set of (1-hop) neighbors of agent i is defined as $\mathcal{N}_i := \{j : (j, i) \in \mathcal{E}\} \cup \{i\}$. The set $\mathcal{N}_{-i} = \mathcal{N}_i \setminus \{i\}$, denotes the set of neighbors of agent i, excluding agent i. The in-degree matrix $\mathcal{D} \in \mathbb{R}^{N\times N}$ is defined as $\mathcal{D} = \mathrm{diag}(d_i)$, where $d_i = \sum_{j\in\mathcal{N}_i} a_{ij}$, and the graph Laplacian matrix $\mathcal{L} \in \mathbb{R}^{N\times N}$ is defined as $\mathcal{L} = \mathcal{D} - \mathcal{A}$. The extended neighborhood set of node i, denoted by $\mathcal{S}_{-i}$, is defined as the set of all other nodes that have a directed path to node i. Formally, $\mathcal{S}_{-i} = \{j \in \mathcal{N} \mid j \neq i \wedge \exists\kappa \leq N, \{j_1, \ldots, j_\kappa\} \subset \mathcal{N} \mid \{(j, j_1), (j_1, j_2), \ldots, (j_\kappa, i)\} \subset 2^{\mathcal{E}}\}$. Let $\mathcal{S}_i = \mathcal{S}_{-i} \cup \{i\}$ and let the edge weights be normalized such that $\sum_j a_{ij} = 1$ for all $i \in \mathcal{N}$. Note that the extended neighborhood subgraphs are nested in the sense that $\mathcal{S}_j \subseteq \mathcal{S}_i$ for all $j \in \mathcal{S}_i$.

B. System Dynamics

Consider a multiagent system with N agents and one leader, where the dynamics of each agent $i \in \mathcal{N}$ are described by the continuous-time linear system

$$\dot{x}_i = A_i x_i + B_i u_i, \tag{1}$$

where $x_i \in \mathbb{R}^n$ is the state, $u_i \in \mathbb{R}^{m_i}$ is the control input, $A_i \in \mathbb{R}^{n\times n}$ is the state transition matrix, and $B_i \in \mathbb{R}^{n\times m_i}$ is the control effectiveness matrix. The leader is referred to as agent 0 with autonomous linear dynamics of the form $\dot{x}_0 = A_0 x_0$, where $x_0 \in \mathbb{R}^n$ denotes the state of the leader and A_0 is the system matrix of the leader. The communication flow between the leader and agent i is characterized by the pinning gain $a_{i0} \geq 0$, where $a_{i0} > 0$ signifies that agent i can observe the leader. The diagonal pinning gain matrix $\mathcal{A}_0 \in \mathbb{R}^{N\times N}$, defined as $\mathcal{A}_0 = \mathrm{diag}([a_{10}, \ldots, a_{N0}])$, represents the information flow between the leader and the followers. The following assumptions describe the class of dynamics and graphs studied in this letter.

Assumption 1: For all $i \in \mathcal{N}$, (A_i, B_i) is controllable.

Assumption 2: The graph has a spanning tree with the leader at the root.

Assumption 2 implies that there exists a directed path from the leader to any node i. If the graph is a directed out-tree with the leader at the root, then the extended neighbors of each agent include all agents between it and the root. On the other hand, if the graph is symmetric, then the extended neighbors of each agent include all other agents.

C. Tracking Error

The control objective is to design a feedback control policy for each agent to track the trajectory of the leader and achieve the desired formation with respect to other agents in the network including the leader. To achieve the desired formation with respect to the leader, let the local neighborhood formation tracking error signals be defined as $e_i := \sum_{j\in\{0\}\cup\mathcal{N}_{-i}} a_{ij} \left((x_i - x_j) - x_{dij}\right)$, where $x_{dij} := x_{di0} - x_{dj0}$ and x_{di0} and x_{dj0} are the constant desired positions of agent i and j with respect to the leader, respectively. The agents reach the desired formation when $x_i(t) - x_j(t) = x_{dij}$ for all $i, j \in \mathcal{N}$. The vectors $\{x_{di0}\}_{i\in\mathcal{N}}$ are unknown to agents not connected to the leader, and the control objective is satisfied when $x_i = x_{di0} + x_0$ for all $i \in \mathcal{N}$. The dynamics of the open-loop neighborhood tracking errors are given as

$$\dot{e}_i = \sum_{j\in\{0\}\cup\mathcal{N}_{-i}} a_{ij}\Big((A_i x_i - A_j x_j) + (B_i u_i - B_j u_j)\Big), \tag{2}$$

for all $i \in \mathcal{N}$. Stacking the error signals in a vector $e = [e_1^\top, \ldots, e_N^\top]^\top \in \mathbb{R}^{nN}$, the error dynamics can be expressed in matrix form as $e = ((\mathcal{L} + \mathcal{A}_0) \otimes I_n)(x - x_d - x_0)$, where $x = [x_1^\top, \ldots, x_N^\top]^\top \in \mathbb{R}^{nN}$, $x_d = [x_{d10}^\top, \ldots, x_{dN0}^\top]^\top \in \mathbb{R}^{nN}$, $x_0 = [x_0^\top, \ldots, x_0^\top]^\top \in \mathbb{R}^{nN}$, and $\otimes$ denotes the Kronecker product. Under Assumption 2, it can be concluded that the matrix $((\mathcal{L} + \mathcal{A}_0) \otimes I_n) \in \mathbb{R}^{nN\times nN}$ is nonsingular [18, Theorem 5]. As a result, $\|e\| = 0$ implies $x_i = x_{di0} + x_0$ for all $i \in \mathcal{N}$, i.e., satisfaction of the control objective.

D. Graph Information Structure

This section encodes the information available to each agent into their measurement matrices. Let $|\mathcal{S}_i|$ denote the cardinality of the set $\mathcal{S}_i$. Assume that $\mathcal{S}_i$ is ordered such that the first element is i and the first $|\mathcal{N}_i|$ elements correspond to direct neighbors. Let $\mathcal{S}_i^o = \{1, \ldots, |\mathcal{S}_i|\}$ be the index set of $\mathcal{S}_i$ and let $\pi_i : \mathcal{S}_i^o \to \mathcal{S}_i$ be a bijective map such that $\pi_i(j)$ (written as π_i^j for brevity) denotes the j-th element of $\mathcal{S}_i$.

We assume that each agent can partially measure its own tracking error and the neighborhood tracking errors of its direct neighbors, as indicated by the output equations $\hat{y}_{ij} = \hat{C}_{ij} e_j \in \mathbb{R}^q$, for $j \in \mathcal{N}_i$, where $\hat{C}_{ij} \in \mathbb{R}^{q\times n}$ is the corresponding measurement matrix. With this notation, the total output measured by each agent, denoted by $y_i := [\hat{y}_{i,\pi_i^1}^\top, \ldots, \hat{y}_{i,\pi_i^{|\mathcal{N}_i|}}^\top]^\top \in \mathbb{R}^{q|\mathcal{N}_i|}$, can be expressed as $y_i = C_i[e_{\pi_i^1}^\top, \ldots, e_{\pi_i^{|\mathcal{S}_i|}}^\top]^\top$, where the output matrices $C_i \in \mathbb{R}^{q|\mathcal{N}_i|\times n|\mathcal{S}_i|}$ are constructed as

$$[C_i]_{j,k} := \begin{cases} \hat{C}_{ij}, & \text{if } j = \pi_i(k) \\ 0_{q\times n}, & \text{otherwise,} \end{cases} \tag{3}$$

for $j = 1, \ldots, |\mathcal{N}_i|$, and $k = 1, \ldots, |\mathcal{S}_i|$.

III. Formation Tracking Control Design

To ensure finiteness of the optimal cost, the optimal control problem is formulated in terms of control errors $\mu_i \in \mathbb{R}^{m_i}$, defined as $\mu_i := \sum_{j\in\{0\}\cup\mathcal{N}_{-i}} a_{ij}(u_i - u_{dij})$, for $i \in \mathcal{N}$ (see [2]). By substituting the desired relative position $x_j + x_{dij}$ into (1), the ideal relative control signals that keep agent i in its desired relative position with respect to agent $j \in \mathcal{N}_{-i}$, denoted by $u_{dij} \in \mathbb{R}^{m_i}$, are computed by solving

$$A_i x_j + A_i x_{dij} + B_i u_{dij} = \dot{x}_j. \tag{4}$$

The assumption below is needed to compute u_{dij} explicitly (see [2, Assumption 2]).

Assumption 3: The control signals u_{dij} satisfying (4) can be expressed *along the desired trajectory* as $u_{dij} = -B_i^\dagger A_{ij} x_j - B_i^\dagger A_i x_{dij} + B_{ij} u_j$, where $A_{ij} := A_i - A_j$, $B_{ij} := B_i^\dagger B_j$, and $B_i^\dagger$ denotes the Moore-Penrose pseudo-inverse of B_i.

Under Assumption 3, the dynamics (2) can be expressed as

$$\dot{e}_i = A_i e_i + B_i \mu_i. \tag{5}$$

In the following section, we treat the control error μ_i in (5) as the design variable, design a control error policy that depends on the tracking errors of the extended neighbors of agent i, and use that to design the actual control signal in (1), see (15).

A. Differential Graphical Games

The objective of each agent is to simultaneously design and utilize a control signal $t \mapsto \mu_i(t)$ online that minimizes the cost functional $J_i := \int_0^\infty r_i\left(e_{\mathcal{N}_i}(\tau), \mu_i(\tau)\right) d\tau$, subject to the tracking error dynamics in (5), where $t \mapsto e_i(t)$ is the solution to the system in (5), starting from initial condition $e_{i,0}$ and under the control signal $t \mapsto \mu_i(t)$, and $e_{\mathcal{N}_i} := \{e_j : j \in \mathcal{N}_i\}$. The local instantaneous cost $r_i : \mathbb{R}^{n|\mathcal{N}_i|} \times \mathbb{R}^{m_i} \to \mathbb{R}_{\geq 0}$ is defined as $r_i(e_{\mathcal{N}_i}, \mu_i) := \sum_{k\in\mathcal{N}_i} \sum_{j\in\mathcal{N}_i} \left(e_k^\top Q_{kj} e_j\right) + \mu_i^\top R_i \mu_i$, where the matrices $Q_{kj} \in \mathbb{R}^{n\times n}$ and $R_i \in \mathbb{R}^{m_i\times m_i}$ are selected such that $e_{\mathcal{N}_i} \mapsto \sum_{k\in\mathcal{N}_i} \sum_{j\in\mathcal{N}_i} \left(e_k^\top Q_{kj} e_j\right)$ is positive definite and $R_i = R_i^\top \succ 0$.

The value function of each agent generally depends on the error states $e_{s_i} := [e_{\pi_i^1}^\top, \ldots, e_{\pi_i^{|\mathcal{S}_i|}}^\top]^\top \in \mathbb{R}^{n|\mathcal{S}_i|}$ of its extended neighborhood, whose dynamics are given by

$$\dot{e}_{s_i} = A_{s_i} e_{s_i} + B_{s_i} \mu_{s_i}, \tag{6}$$

where $A_{s_i} := \text{blkdiag}(A_{\pi_i^1}, \ldots, A_{\pi_i^{|\mathcal{S}_i|}}) \in \mathbb{R}^{n|\mathcal{S}_i|\times n|\mathcal{S}_i|}$, $B_{s_i} := \text{blkdiag}(B_{\pi_i^1}, \ldots, B_{\pi_i^{|\mathcal{S}_i|}}) \in \mathbb{R}^{n|\mathcal{S}_i|\times \sum_{j\in\mathcal{S}_i} m_j}$, and $\mu_{s_i} = [\mu_{\pi_i^1}^\top, \ldots, \mu_{\pi_i^{|\mathcal{S}_i|}}^\top]^\top \in \mathbb{R}^{\sum_{j\in\mathcal{S}_i} m_j}$ is the stacked vector of control policies corresponding to the extended neighbors of agent i. The notation $\text{blkdiag}(\cdot)$ denotes the block diagonal concatenation operation. The value function for agent i is denoted by $V_i : \mathbb{R}^{n|\mathcal{S}_i|} \times \mathbb{R}^{m_i} \times \mathbb{R}^{\sum_{j\in\mathcal{S}_{-i}} m_j} \to \mathbb{R}_{\geq 0}$ and is defined as

$$V_i\left(e_{s_i}, \mu_i, \mu_{s_{-i}}\right) := \int_t^\infty r_i\left(e_{s_i}(\tau), \mu_{s_i}(e_{s_i}(\tau))\right) d\tau, \tag{7}$$

which is the total cost-to-go for agent i given control policies $\left\{\mu_j : \mathbb{R}^{n|\mathcal{S}_j|} \to \mathbb{R}^{m_j}\right\}_{j\in\mathcal{S}_i}$, where $\mu_{s_{-i}}$ is the set of control policies of the extended neighbors of agent i, excluding agent i itself, i.e., $\mu_{s_{-i}} = \{\mu_j : j \in \mathcal{S}_i\} \setminus \{\mu_i\}$.

Since each agent tries to minimize their own cost function, the optimization problem is a multi-objective optimization problem. To define network-level optimality, we utilize the concept of an FNE. The tuple of policies $\{\mu_j^* : \mathbb{R}^{n|\mathcal{S}_j|} \to \mathbb{R}^{m_j}\}_{j\in\mathcal{S}_i}$ constitutes an FNE solution within the subgraph $\mathcal{S}_i$ if the value functions in (7) satisfy [2]

$$V_j^*(e_{s_j}) := V_j\left(e_{s_j}, \mu_j^*, \mu_{s_{-j}}^*\right) \leq V_j\left(e_{s_j}, \mu_j, \mu_{s_{-j}}^*\right), \tag{8}$$

for all $j \in \mathcal{S}_i$, for all $e_{s_j} \in \mathbb{R}^{n|\mathcal{S}_j|}$, and for all admissible policies $\{\mu_j\}_{j\in\mathcal{S}_i}$. To characterize the FNE solution, let $W_i := \sum_{j\in\mathcal{S}_{-i}} E_{ij} B_j R_j^{-1} B_j^\top E_{jj}^\top P_{s_j} \Lambda_{ji}$, where $E_{ij} := \nabla_{e_j} e_{s_i} \in \mathbb{R}^{n|\mathcal{S}_i|\times n}$ and $\Lambda_{ji} \in \mathbb{R}^{n|\mathcal{S}_j|\times n|\mathcal{S}_i|}$ is a matrix satisfying $e_{s_j} = \Lambda_{ji} e_{s_i}$, whose (k,l)-th block is defined as

$$[\Lambda_{ji}]_{k,l} = \begin{cases} I_n, & \text{if } k \in \mathcal{S}_j \wedge l = \pi_i(k) \\ 0_{n\times n}, & \text{otherwise.} \end{cases} \tag{9}$$

Such a matrix Λ_{ji} exists since $\mathcal{S}_j \subseteq \mathcal{S}_i$.

Theorem 1: If the algebraic Riccati equations (AREs)

$$P_{s_i}(A_{s_i} - W_i) + (A_{s_i} - W_i)^\top P_{s_i} - P_{s_i} E_{ii} B_i R_i^{-1} B_i^\top E_{ii}^\top P_{s_i} + Q_{s_i} = 0, \tag{10}$$

admit symmetric positive definite solutions P_{s_i} for all $i \in \mathcal{N}$, then the control policies

$$\mu_i^*(e_{s_i}) = -K_{s_i} e_{s_i}, \tag{11}$$

constitute an FNE solution to the differential graphical game, where $K_{s_i} := R_i^{-1} B_i^\top E_{ii}^\top P_{s_i} \in \mathbb{R}^{m_i\times n|\mathcal{S}_i|}$.

Proof: Using arguments similar to [2, Theorem 1], it can be shown that if the value functions for all agents $i \in \mathcal{N}$ are continuously differentiable and if an FNE solution exists, then the FNE value functions are solutions of the coupled Hamilton-Jacobi (HJ) equations

$$\nabla_{e_i} V_i^*(e_{s_i})\left(A_i e_i + B_i \mu_i^*(e_{s_i})\right) + \sum_{j\in\mathcal{S}_{-i}} \nabla_{e_j} V_i^*(e_{s_i})\left(A_j e_j + B_j \mu_j^*(e_{s_j})\right) + \sum_{k\in\mathcal{N}_i} \sum_{j\in\mathcal{N}_i} \left(e_k^\top Q_{kj} e_j\right) + \mu_i^{*\top}(e_{s_i}) R_i \mu_i^*(e_{s_i}) = 0, \tag{12}$$

for all $e_{s_i} \in \mathbb{R}^{n|\mathcal{S}_i|}$. The FNE policy $\mu_i^* : \mathbb{R}^{n|\mathcal{S}_i|} \to \mathbb{R}^{m_i}$ for agent i is then given by $\mu_i^*(e_{s_i}) = -\frac{1}{2} R_i^{-1} B_i^\top \nabla_{e_i} V_i^*(e_{s_i})^\top$.

Since the dynamics in (6) are linear, the FNE value functions are of the form $V_i^*(e_{s_i}) = e_{s_i}^\top P_{s_i} e_{s_i}$, where the matrix $P_{s_i} \in \mathbb{R}^{n|\mathcal{S}_i|\times n|\mathcal{S}_i|}$ satisfies $P_{s_i} = P_{s_i}^\top \succ 0$ and $\nabla_{e_i} V_i^*(e_{s_i}) = 2 e_{s_i}^\top P_{s_i} E_{ii}$ (see [19, Chapter 6]). Let $Q_{s_i} := \sum_{k\in\mathcal{N}_i} \sum_{j\in\mathcal{N}_i} E_{ik} Q_{kj} E_{ij}^\top$. Substituting the gradient of V_i^* into the HJ equations in (12) yields

$$e_{s_i}^\top \Big[P_{s_i} A_{s_i} + A_{s_i}^\top P_{s_i} - P_{s_i} E_{ii} B_i R_i^{-1} B_i^\top E_{ii}^\top P_{s_i} - 2\sum_{j\in\mathcal{S}_{-i}} P_{s_i} E_{ij} B_j R_j^{-1} B_j^\top E_{jj}^\top P_{s_j} \Lambda_{ji} + Q_{s_i}\Big] e_{s_i} = 0. \tag{13}$$

The HJ equation in (13) holds if and only if P_{s_i} satisfies the ARE in (10). Thus, if the ARE in (10) is satisfied for all $i \in \mathcal{N}$, then the FNE strategies are given by (11). ■

Remark 1: In the simulation example in Section VI, the coupled AREs in (10) are solved using the method developed in [20]. For a discussion on the existence of solutions to coupled Riccati equations, see [21, Chapter 6].

Implementation of the control policy in (11), requires each agent in the network to know the system matrices A_j, B_j, Q_j, and R_j of their extended neighbors to compute the positive definite matrix P_{s_i} and the control gain K_{s_i}. The following assumption formalizes this requirement.

Assumption 4: The matrices A_j, B_j, Q_j, and R_j are known to each agent i for all $j \in \mathcal{S}_i$.

Remark 2: The development in [8] requires only local information, i.e., knowledge of A_j, B_j, Q_j, and R_j of only the direct neighbors, but utilizes a much stronger assumption that the optimal value function depends only on the error-states of the direct neighbors. The method in [9] develops conditions on the cost function that decentralize the resulting controllers, however, an assumption similar to Assumption 4 is needed for computation of the controllers. Similarly, the method in [3] uses only local information but prioritizes stabilizing solutions over approximate FNE ones. The SED-based approach in [10] relies on decaying influence, but any finite truncation results in loss of optimality. In summary, all existing approaches to solve this problem either use stronger assumptions than Assumption 4 regarding what the agents know about each other or compromise near-optimality in favor of stabilization. Assumption 4, while restrictive, provides an alternative formulation of distributed optimal control that explicitly considers the influence of neighbors of agent i beyond its direct neighbors.

By stacking the local control policies $\mu_j^*(e_{s_i})$ from (11) for each $j \in \mathcal{S}_i$, the extended neighborhood control vector in (6) is given by $\mu_{s_i}^*(e_{s_i}) = -\mathcal{K}_{s_i} e_{s_i}$, where $\mathcal{K}_{s_i} := [\Lambda_{\pi_i^1 i}^\top K_{s_{\pi_i^1}}^\top, \ldots, \Lambda_{\pi_i^{|\mathcal{S}_i|} i}^\top K_{s_{\pi_i^{|\mathcal{S}_i|}}}^\top]^\top \in \mathbb{R}^{\sum_{j\in\mathcal{S}_i} m_j \times n|\mathcal{S}_i|}$. Since the extended neighborhood tracking error e_{s_i} is unknown to agent i, it needs to be estimated using the output, y_i. While decentralized observer techniques like [17], [22] may not allow full-state reconstruction, especially when the graph $\mathcal{G}$ is neither strongly connected nor composed of independent strongly connected components, it is still possible to reconstruct e_{s_i} using y_i. In Section IV, we develop local observers to generate estimates $\hat{e}_{s_i} := [\hat{e}_{\pi_i^1}^\top, \ldots, \hat{e}_{\pi_i^{|\mathcal{S}_i|}}^\top]^\top \in \mathbb{R}^{n|\mathcal{S}_i|}$ where $\hat{e}_{\pi_i^j}$ is the estimated local tracking error of the j-th extended neighbor of the i-th agent, constructed by the i-th agent. The estimates $\hat{e}_{s_i}$ are then used to compute an approximate FNE policy $\hat{\mu}_i : \mathbb{R}^{n|\mathcal{S}_i|} \to \mathbb{R}^{m_i}$ given by

$$\hat{\mu}_i(\hat{e}_{s_i}) = -K_{s_i}\hat{e}_{s_i}, \tag{14}$$

To infer the approximate control policy u_i for agent i from $\hat{\mu}_i$, let the matrix $\sigma_i \in \mathbb{R}^{m_i \times m_i}$ be defined as $\sigma_i := \sum_{j\in\{0\}\cup\mathcal{N}_{-i}} a_{ij} I_{m_i}$ and let $\hat{z}_i = \sum_{j\in\{0\}\cup\mathcal{N}_{-i}} a_{ij} B_i^\dagger A_{ij}\hat{x}_j + B_i^\dagger A_i x_{dij} \in \mathbb{R}^{m_i}$. Using this notation, the control policy for agent i in (1) can be expressed in the explicit form

$$u_i(t) = -\sigma_i^{-1} K_{s_i}\hat{e}_{s_i} - \sigma_i^{-1}\hat{z}_i. \tag{15}$$

The following section introduces distributed state observers to estimate the extended neighborhood tracking errors required by the approximate FNE control policy in (15).

IV. Distributed observer Design

Motivated by the distributed observers in [14], [17], the observer utilizes a consensus term of the form $\beta_{ji}\hat{e}_{s_j} - \beta_{ij}\hat{e}_{s_i}$, where $\beta_{ij} \in \mathbb{R}^{n|\mathcal{S}_i|\times n|\mathcal{S}_i|}$ and $\beta_{ji} \in \mathbb{R}^{n|\mathcal{S}_i|\times n|\mathcal{S}_j|}$ are matrices that select error estimates corresponding to common extended neighbors of agent i and j. The difference $\beta_{ji}\hat{e}_{s_j} - \beta_{ij}\hat{e}_{s_i}$ drives the error estimates of all direct neighbors in the network to consensus by allowing each agent use the error estimates $\hat{e}_{s_j}$ of its direct neighbors to improve the estimate $\hat{e}_{s_i}$ of its extended neighborhood tracking error.

To define β_{ij} and β_{ji} for general networks, let $\mathcal{S}_{ij}^o$ be the index set of the shared extended neighbors between agents i and j, i.e., $\mathcal{S}_{ij}^o = \mathcal{S}_i^o \cap \mathcal{S}_j^o$, for $i, j \in \mathcal{N}_i$ and $j \neq i$. Let $\pi_{ij} : \mathcal{S}_{ij}^o \to \mathcal{S}_i$ and $\pi_{ji} : \mathcal{S}_{ij}^o \to \mathcal{S}_j$ be bijections that map elements of $\mathcal{S}_{ij}^o$ to their positions in $\mathcal{S}_i$ and $\mathcal{S}_j$, respectively, such that $\pi_{ij}(1) = i$ and $\pi_{ji}(1) = j$. The matrix β_{ij} can now be defined blockwise as

$$[\beta_{ij}]_{k,l} = \begin{cases} I_n, & \text{if } k = l = \pi_{ij}(s) \text{ for } s \in \mathcal{S}_{ij}^o, \\ 0_n, & \text{otherwise,} \end{cases} \tag{16}$$

and the matrix β_{ji} can be expressed blockwise as

$$[\beta_{ji}]_{k,l} = \begin{cases} I_n, & \text{if } k = \pi_{ij}(s) \wedge l = \pi_{ji}(s) \text{ for } s \in \mathcal{S}_{ij}^o, \\ 0_n, & \text{otherwise,} \end{cases} \tag{17}$$

for all $k \in \mathcal{S}_i$ and $l \in \mathcal{S}_j$, where $[\beta_{ij}]_{k,l}$ denotes the $n \times n$ block in the k, l position of β_{ij}.

Since the pair (A_{s_i}, C_i) may not be observable, this paper employs an observer design approach using Kalman observability decomposition similar to [14], [17], [22], and [23]. For all $i \in \mathcal{S}_i$, let $\nu_i \in \mathbb{Z}^+$ be the dimension of the observable subspace of the pair (A_{s_i}, C_i) denoted by $\mathrm{Im}(\mathcal{O})$, such that $\mathrm{rank}(\mathcal{O}_i) = \nu_i$, where $\mathcal{O}_i := [C_i, C_i A_{s_i}, \ldots, C_i A_{s_i}^{(n|\mathcal{S}_i|-1)}]^\top$ is the observability matrix associated with the pair (A_{s_i}, C_i). Then, the unobservable subspace $\mathrm{Ker}(\mathcal{O}_i)$ is of dimension $\rho_i := n|\mathcal{S}_i| - \nu_i$, and satisfies $\mathrm{Ker}(\mathcal{O}_i)^\perp = \mathrm{Im}(\mathcal{O}_i^\top)$[1]. For all $i \in \mathcal{S}_i$, let $\Sigma_{u_i} \in \mathbb{R}^{n|\mathcal{S}_i|\times\rho_i}$ be a matrix whose columns form an orthonormal basis of $\mathrm{Ker}(\mathcal{O}_i)$ such that $\mathrm{Im}(\Sigma_{u_i}) = \ker(\mathcal{O}_i)$, and let $\Sigma_{o_i} \in \mathbb{R}^{n|\mathcal{S}_i|\times\nu_i}$ be a matrix whose columns form an orthonormal basis for $\mathrm{Im}(\mathcal{O}_i^\top)$ such that $\mathrm{Im}(\Sigma_{o_i}) = \mathrm{Im}(\mathcal{O}_i^\top)$. For each $i \in \mathcal{N}$, define an orthogonal transformation matrix $\Sigma_i := [\Sigma_{o_i}, \Sigma_{u_i}] \in \mathbb{R}^{n|\mathcal{S}_i|\times n|\mathcal{S}_i|}$ such that $\Sigma_i^\top\Sigma_i = I_{n|\mathcal{S}_i|}$ for all $i \in \mathcal{S}_i$. Using Kalman observability decomposition, the matrices A_{s_i} and C_i can be expressed as $\Sigma_i^\top A_{s_i}\Sigma_i = [\bar{A}_i, 0_{\nu_i\times\rho_i}; 0_{\rho_i\times\nu_i}, \hat{A}_i]$ and $C_i\Sigma_i = [\bar{C}_i, 0_{q|\mathcal{N}_i|\times\rho_i}]$, respectively, where $\bar{A}_i \in \mathbb{R}^{\nu_i\times\nu_i}$, $\hat{A}_i \in \mathbb{R}^{\rho_i\times\rho_i}$, $\bar{C}_i \in \mathbb{R}^{q|\mathcal{N}_i|\times\nu_i}$ are defined such that $(\bar{A}_i, \bar{C}_i)$ is an observable pair, $\bar{A}_i$ and $\hat{A}_i$ are skew-symmetric matrices, $\Sigma_{o_i}^\top A_{s_i}\Sigma_{o_i} = \bar{A}_i$, and $\Sigma_{u_i}^\top A\Sigma_{u_i} = \hat{A}_i$ [23, Lemma 2]. The observer is designed as

$$\begin{aligned}\dot{\hat{e}}_{s_i} &= A_{s_i}\hat{e}_{s_i} + B_{s_i}\hat{\mu}_{s_i}(\hat{e}_{s_i}) + G_i\left(y_i - C_i\hat{e}_{s_i}\right) \\ &\quad + \gamma M_i {\textstyle\sum_{i\in\mathcal{N}_i}} a_{ij}\left(\beta_{ji}\hat{e}_{s_j} - \beta_{ij}\hat{e}_{s_i}\right),\end{aligned} \tag{18}$$

where $\hat{\mu}_{s_i}(\hat{e}_{s_i}) = -\mathcal{K}_{s_i}\hat{e}_{s_i} \in \mathbb{R}^{\sum_{j\in\mathcal{S}_i} m_j}$, $G_i \in \mathbb{R}^{n|\mathcal{S}_i|\times q|\mathcal{N}_i|}$ and $M_i \in \mathbb{R}^{n|\mathcal{S}_i|\times n|\mathcal{S}_i|}$ are observer gains matrices designed as $G_i := [\bar{G}_i^\top, \ 0_{q|\mathcal{N}_i|\times\rho_i}]^\top$ and $M_i = \Sigma_i[0_{\nu_i\times\nu_i}, 0_{\nu_i\times\rho_i}; 0_{\rho_i\times\nu_i}, I_{\rho_i}]\Sigma_i^\top$, respectively, where $\bar{G}_i \in \mathbb{R}^{\nu_i\times q|\mathcal{N}_i|}$ satisfies

$$(\bar{A}_i - \bar{G}_i\bar{C}_i)^\top\Theta_i + \Theta_i(\bar{A}_i - \bar{G}_i\bar{C}_i) = -\alpha I_{\nu_i}, \tag{19}$$

[1]For any matrix $A \in \mathbb{R}^{m\times n}$, $\mathrm{Im}(A) = \{y \in \mathbb{R}^m \mid \exists x \in \mathbb{R}^n,\ Ax = y\}$, $\ker(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$.

for some symmetric positive definite matrix $\Theta_i \in \mathbb{R}^{\nu_i \times \nu_i}$ and constant $\alpha \in \mathbb{R}_{>0}$, and $\gamma \in \mathbb{R}_{>0}$ is a observer gain. In real-world robotic systems, each agent in the network needs proprioceptive sensors (e.g., IMUs or encoders) and exteroceptive sensors (e.g., GPS, cameras, or LiDAR) to achieve state observability and communication equipment to query partial tracking errors from its direct neighbors.

Let the estimation errors be defined as $\tilde{e}_{s_i} := e_{s_i} - \hat{e}_{s_i}$. Since the selection matrices in (16) and (17) both satisfy $\beta_{ji} e_{s_j} = \beta_{ij} e_{s_i}$, the estimation error dynamics can be expressed as $\dot{\tilde{e}}_{s_i} = (A_{s_i} - G_i C_i)\,\tilde{e}_{s_i} + \gamma M_i \sum_{i \in \mathcal{N}_i} a_{ij} \left(\beta_{ji}\tilde{e}_{s_j} - \beta_{ij}\tilde{e}_{s_i}\right)$. Let $\zeta_{s_i} = \Sigma_{o_i}^\top \tilde{e}_{s_i}$, $\xi_{s_i} = \Sigma_{u_i}^\top \tilde{e}_{s_i}$. Then the estimation error dynamics can be split into

$$\dot{\zeta}_{s_i} = \left(\bar{A}_i - \bar{G}_i \bar{C}_i\right)\zeta_{s_i}, \tag{20}$$

$$\dot{\xi}_{s_i} = \hat{A}_i \xi_{s_i} - \gamma \Sigma_{u_i}^\top \sum_{i \in \mathcal{N}_i} l_{ij} \beta_{ji} \left(\Sigma_{u_j}\xi_{s_j} + \Sigma_{o_j}\zeta_{s_j}\right), \tag{21}$$

where l_{ij} is the (i,j)-th entry of the Laplacian matrix $\mathcal{L}$.

V. Stability Analysis

To facilitate the analysis, let $\zeta_s := [\zeta_{s_1}^\top, \ldots, \zeta_{s_{|\mathcal{S}_p|}}^\top]^\top$, $\xi_s := [\xi_{s_1}^\top, \ldots, \xi_{s_{|\mathcal{S}_p|}}^\top]^\top$, $\Sigma_o := [\Sigma_{o_1}^\top, \ldots, \Sigma_{o_{|\mathcal{S}_p|}}^\top]^\top$, $\Sigma_u := [\Sigma_{u_1}^\top, \ldots, \Sigma_{u_{|\mathcal{S}_p|}}^\top]^\top$, $\bar{A} = \operatorname{blkdiag}(\bar{A}_1, \ldots, \bar{A}_{|\mathcal{S}_p|})$, $\bar{G} = \operatorname{blkdiag}(\bar{G}_1, \ldots, \bar{G}_{|\mathcal{S}_p|})$, $\bar{C} = \operatorname{blkdiag}(\bar{C}_1, \ldots, \bar{C}_{|\mathcal{S}_p|})$, $\hat{A} = \operatorname{blkdiag}(\hat{A}_1, \ldots, \hat{A}_{|\mathcal{S}_p|})$, $\beta_s = \operatorname{blkdiag}(\beta_1, \ldots, \beta_{|\mathcal{S}_p|})$, where $\beta_i = \operatorname{blkdiag}(\beta_{i\pi_i^1}, \ldots, \beta_{i\pi_i^{|\mathcal{S}_i|}})$. The dynamics in (20) and (21) can then be expressed as

$$\dot{\zeta}_s = \left(\bar{A} - \bar{G}\bar{C}\right)\zeta_s, \tag{22}$$

$$\dot{\xi}_s = \left(\hat{A} - \gamma \Sigma_u^\top \Delta_s \Sigma_u\right)\xi_s - \gamma \Sigma_u^\top \Delta_s \Sigma_o \zeta_s, \tag{23}$$

where $\Delta_s = (\mathcal{L} \otimes I_n)\,\beta_s$ and $\lambda_{\min}(\Delta_s) > 0$. Consider a set of extended neighbors $\mathcal{S}_p$ corresponding to the p-th agent and define the concatenated state vector $Z_p := [e_{s_p}^\top, \zeta_{s_p}^\top, \xi_{s_p}^\top]^\top \in \mathbb{R}^{2n|\mathcal{S}_p|}$. Let $V_p : \mathbb{R}^{2n|\mathcal{S}_p|} \to \mathbb{R}$ be a candidate Lyapunov function for (6), (20), and (21), defined as

$$V_p(Z_p) = \sum_{i \in \mathcal{S}_p} \left(V_i^*(e_{s_i}) + \zeta_{s_i}^\top \Theta_i \zeta_{s_i} + \xi_{s_i}^\top \Gamma_i \xi_{s_i}\right), \tag{24}$$

which satisfies the inequality $\underline{v}_p(\|Z_p\|) \leq V_p(Z_p) \leq \overline{v}_p(\|Z_p\|)$, where $\underline{v}_p, \overline{v}_p : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ are class $\mathcal{K}_\infty$ functions and $\Gamma_i \in \mathbb{R}^{n|\mathcal{S}_i| \times n|\mathcal{S}_i|}$ is a positive definite gain matrix. The convergence properties of the closed-loop system are summarized in the following theorem.

Theorem 2: If the coupled AREs in (10) admit positive definite solutions $\{P_{s_i} \succ 0\}_{i \in \mathcal{S}_p}$, assumptions 1–4 hold, the observer gains $\{G_i\}_{i \in \mathcal{S}_p}$ are selected such that they satisfy (19), γ is selected such that $\dot{\xi}_s = \left(\hat{A} - \gamma \Sigma_u^\top \Delta_s \Sigma_u\right)\xi_s$ is asymptotically stable, and the sufficient conditions in (28) are satisfied, then the closed loop system in (6), (20), and (21) is globally exponentially stable.

Proof: The Lie derivative of the candidate Lyapunov function in (24) along the flow of (6), (20), and (21), and under the approximate FNE policy in (14) is given by

$$\begin{aligned} \dot{V}_p(Z_p) &= \sum_{i \in \mathcal{S}_p} \nabla_{e_{s_i}} V_i^*(e_{s_i}) \Big(A_{s_i} e_{s_i} + B_{s_i} \hat{\mu}_{s_i}(\hat{e}_{s_i})\Big) \\ &+ \sum_{i \in \mathcal{S}_p} 2\xi_{s_i}^\top \Gamma_i \big(\hat{A}_i \xi_{s_i} - \gamma \Sigma_{u_i}^\top \sum_{j \in \mathcal{N}_i} l_{ij}\beta_{ji} \left(\Sigma_{u_j}\xi_{s_j} + \Sigma_{o_j}\zeta_{s_j}\right)\big) \\ &+ \sum_{i \in \mathcal{S}_p} \zeta_{s_i}^\top \left((\bar{A}_i - \bar{G}_i\bar{C}_i)^\top \Theta_i + \Theta_i(\bar{A}_i - \bar{G}_i\bar{C}_i)\right)\zeta_{s_i}. \end{aligned} \tag{25}$$

Substituting (12), (10), (14), (19), and (23), and using the graph Laplacian properties to handle the cross terms in the consensus term, the Lie derivative in (25) can be bounded as

$$\begin{aligned} \dot{V}_p(Z_p) \leq &-e_s Q_s e_s - \alpha \zeta_s^\top \zeta_s - 2\gamma \xi_s^\top \Sigma_u^\top \Delta_s \Sigma_o \zeta_s \\ &+ 2e_s^\top P_s B_s \mathcal{K}_s(\Sigma_u \xi_s + \Sigma_o \zeta_s) \\ &+ \xi_s^\top \left((\hat{A} - \gamma\Sigma_u^\top \Delta_s \Sigma_u)^\top \Gamma + \Gamma(\hat{A} - \gamma \Sigma_u^\top \Delta_s \Sigma_u)\right)\xi_s, \end{aligned} \tag{26}$$

where $e_s = [e_{s_1}^\top, \ldots, e_{s_{|\mathcal{S}_p|}}^\top]^\top$, $P_s = \operatorname{blkdiag}(P_{s_1}, \ldots, P_{s_{|\mathcal{S}_p|}})$, $Q_s = \operatorname{blkdiag}(Q_{s_1}, \ldots, Q_{s_{|\mathcal{S}_p|}})$, $\mathcal{K}_s = \operatorname{blkdiag}(\mathcal{K}_{s_1}, \ldots, \mathcal{K}_{s_{|\mathcal{S}_p|}})$, and $\Gamma = \operatorname{blkdiag}(\Gamma_1, \ldots, \Gamma_{|\mathcal{S}_p|})$. As shown in [22, Lemma 4], γ can be selected sufficiently large such that $\dot{\xi}_s = \left(\hat{A} - \gamma \Sigma_u^\top \Delta_s \Sigma_u\right)\xi_s$ is asymptotically stable. For such a γ, there exists a matrix $\Gamma \succ 0$ and a scalar $\delta > 0$ satisfying $(\hat{A} - \gamma\Sigma_u^\top \Delta_s \Sigma_u)^\top \Gamma + \Gamma(\hat{A} - \gamma\Sigma_u^\top \Delta_s \Sigma_u) < -\delta I_{n\chi}$, where $\chi = \sum_{j \in \mathcal{S}_i} |\mathcal{S}_j|$. Applying completion of squares, the triangle inequality, and the Cauchy-Schwarz inequality, the derivative is bounded as

$$\begin{aligned} \dot{V}_p(Z_p) \leq &-\tfrac{\lambda_{\min}(Q_s)}{3} \sum_{i \in \mathcal{S}_p} \|e_{s_i}\|^2 - \tfrac{\alpha}{3} \sum_{i \in \mathcal{S}_p} \|\zeta_{s_i}\|^2 \\ &- \tfrac{\delta}{2} \sum_{i \in \mathcal{S}_p} \|\xi_{s_i}\|^2 - \left(\tfrac{\alpha}{3} - \tfrac{3\kappa^2 \overline{\Sigma}_o^2}{\lambda_{\min}(Q_s)}\right) \sum_{i \in \mathcal{S}_p} \|\zeta_{s_i}\|^2 \\ &- \left(\tfrac{\delta}{2} - \tfrac{3\kappa^2 \overline{\Sigma}_u^2}{\lambda_{\min}(Q_s)} - \tfrac{3\gamma^2 \overline{\Sigma}_o^2 \overline{\Delta_s}^2 \overline{\Sigma}_u^2}{\alpha}\right) \sum_{i \in \mathcal{S}_p} \|\xi_{s_i}\|^2, \end{aligned} \tag{27}$$

where $\kappa = \|P_s B_s \mathcal{K}_s\|$, $\overline{\Sigma}_o = \|\Sigma_o\|$, $\overline{\Sigma}_u = \|\Sigma_u\|$, and $\overline{\Delta_s} = \|\Delta_s\|$. Hence, provided the gain conditions

$$\frac{\alpha}{3} > \frac{3\kappa^2 \overline{\Sigma}_o^2}{\lambda_{\min}(Q_s)} \text{ and } \frac{\delta}{2} > \frac{3\kappa^2 \overline{\Sigma}_u^2}{\lambda_{\min}(Q_s)} + \frac{3\gamma^2 \overline{\Sigma}_o^2 \underline{c} \Delta_s{}^2 \overline{\Sigma}_u^2}{\alpha}, \tag{28}$$

are satisfied, it can be concluded that $\dot{V}_p(Z_p) \leq -\underline{c}\|Z_p\|^2$, for all $Z_p \in \mathbb{R}^{n|\mathcal{S}_p|}$, where $\underline{c} = \frac{1}{2}\min\{\frac{\lambda_{\min}(Q_s)}{3}, \frac{\alpha}{3}, \frac{\delta}{2}\}$. Invoking [24, Theorem 4.10], it can be concluded that the closed-loop system is globally exponentially stable. ■

VI. Simulation Results

As a numerical example, consider a multiagent system with five agents and one leader connected by a directed graph shown in Figure 1. The dynamics of each agent are described by the continuous-time linear system in (1), where $A_i = [0.5, 1.5; 2, -2]$ and $B_i = [2, 0; 0, 2]$ for all $i = 1, \ldots, 5$, with the output matrix for each agent of the form in (3) where $\hat{C}_{ij} = [1,\ 0]^\top$ for all i and j.

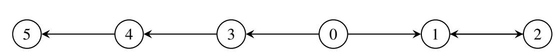

Fig. 1. Communication Topology

The agents start at the origin, and their final desired relative positions are given by $x_{d_{12}} = [-0.5, 1]^\top$, $x_{d_{21}} = [0.5, -1]^\top$, $x_{d_{43}} = [0.5, 1]^\top$, and $x_{d_{53}} = [-1, 1]^\top$. The relative positions are designed such that the final desired formation is a pentagon with the leader node at the center. The leader traverses a sinusoidal trajectory $x_0(t) = [2\sin(t), 2\sin(t) + 2\cos(t)]^\top$. The desired positions of agents 1 and 3 with respect to the leader are $x_{d10} = [-1, 0]^\top$ and $x_{d30} = [0.5, -1]^\top$, respectively. Each agent's initial estimates of their extended neighborhood tracking errors are selected as $\hat{e}_{s_1} = [-0.5, 1.5, 0.5, -1.0]^\top$ for agent 1, $\hat{e}_{s_2} = [-1.0, -1.5, 0.5, 2.0]^\top$ for agent 2, $\hat{e}_{s_3} = [1.5, -2.0]^\top$ for agent 3, $\hat{e}_{s_4} = [-2.0, 1.5, -1.0, 2.5]^\top$ for agent 4, and $\hat{e}_{s_5} = [2.5, -3.0, -2.0, 2.5, -1.5, 3.5]^\top$ for agent 5. The state penalty matrix for each agent Q_{s_i} is constructed from a base matrix $Q_i = I_{2\times 2}$, where diagonal blocks of Q_{s_i} are Q_i, and the off-diagonal blocks are $0.5Q_i$ so that $(Q_{s_i})_{ij} = (Q_{s_i})_{ji}$ for $i, j \in \mathcal{N}_i$. The control penalty matrix is selected as $R_i = I_{2\times 2}$. The observer gain G_i is selected to satisfy (19) with $\alpha = 3$ and $\gamma = 5$ for each agent.

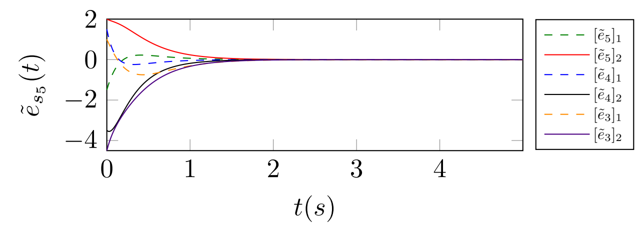

Fig. 2. Trajectories of the estimation error for agent **5** and its extended neighbors.

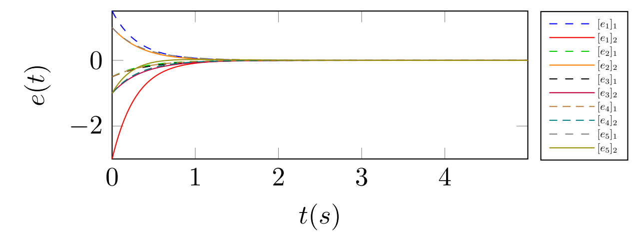

Fig. 3. Trajectories of formation tracking errors for each agent.

A. Discussion

As an example, Fig. 2 shows that the trajectories of the actual extended neighborhood tracking errors and their estimates converge for Agent 5, indicating the effectiveness of the designed observer in (18). Observe that even though Agent 3 is not a direct neighbor of Agent 5, the developed observer in (18) enables Agent 5 to estimate the state of Agent 3. As shown in Fig. 3, the tracking error converges to zero, demonstrating convergence to the desired formation and trajectory. Note that Agents 2, 4, and 5, even without a communication link to the leader and knowledge of their relative positions with respect to the leader, achieve the desired formation using the control policy in (15). The simulation results show that agents can achieve formation without direct communication with the leader and with only partial knowledge of the states of their directed neighbors, despite the graph lacking strong connectivity or independent strongly connected subgraphs.

VII. Conclusion

This letter presents an output feedback game-theoretic framework to achieve simultaneous distributed formation tracking in linear multiagent systems with local communication. Distributed observers are employed by each agent to obtain estimates of the error states of their respective extended neighbors using error state information of their direct neighbors. The estimates are used to compute the FNE policies.

The developed approach requires the somewhat strict assumption that all extended neighbors know the dynamics and the cost functions of each other. This assumption is utilized in the observer design to compute the control inputs of the extended neighbors. While such computation requires solution of Riccati equations in the linear case studied here, extension to nonlinear systems is challenging due to the need to solve HJ equations. The development of observers that do not require such knowledge is a topic for future research.

References

[1] L. Bakule, "Decentralized control: An overview," *Annu. Rev. Control*, vol. 32, no. 1, pp. 87–98, 2008.

[2] R. Kamalapurkar, J. R. Klotz, P. Walters, and W. E. Dixon, "Model-based reinforcement learning in differential graphical games," *IEEE Trans. Control Netw. Syst.*, vol. 5, no. 1, pp. 423–433, Mar. 2018.

[3] D. Cappello and T. Mylvaganam, "Distributed differential games for control of multi-agent systems," *IEEE Trans. Control Netw. Syst.*, vol. 9, no. 2, pp. 635–646, 2022.

[4] S. Shin, Y. Lin, G. Qu, A. Wierman, and M. Anitescu, "Near-optimal distributed linear-quadratic regulator for networked systems," *SIAM J. Control Optim.*, vol. 61, no. 3, pp. 1113–1135, 2023.

[5] G. Leitmann, *Cooperative and non-cooperative many player differential games*. Springer, 1974.

[6] K. G. Vamvoudakis, "Online solution of nonlinear two-player zero-sum games using synchronous policy iteration," *Int. J. Robust Nonlinear Control*, vol. 22, no. 13, pp. 1460–1483, 2012.

[7] D. Vrabie and F. L. Lewis, "Integral reinforcement learning for online computation of feedback Nash strategies of nonzero-sum differential games," in *Proc. IEEE Conf. Decis. Control*, 2010, pp. 3066–3071.

[8] K. G. Vamvoudakis and F. L. Lewis, "Multi-player non-zero-sum games: online adaptive learning solution of coupled Hamilton-Jacobi equations," *Automatica*, vol. 47, pp. 1556–1569, 2011.

[9] M. Liu, Y. Wan, V. G. Lopez, F. L. Lewis, G. A. Hewer, and K. Estabridis, "Differential graphical game with distributed global Nash solution," *IEEE Trans. Control Netw. Syst.*, vol. 8, no. 3, pp. 1371–1382, 2021.

[10] R. C. Zhang, W. Li, and N. Li, "On the optimal control of network LQR with spatially-exponential decaying structure," in *Proc. Am. Control Conf.*, 2023, pp. 1775–1780.

[11] A. G. O. Mutambara, *Decentralized Estimation and Control for Multisensor Systems*, 1st ed. Routledge, 1998.

[12] P. Hilgers and C. Ament, "Distributed and decentralised estimation of non-linear systems," in *Proc. IEEE Int. Conf. Control Appl.*, 2010, pp. 328–333.

[13] S. Park and N. C. Martins, "Necessary and sufficient conditions for the stabilizability of a class of LTI distributed observers," in *Proc. IEEE Conf. Decis. Control*, 2012, pp. 7431–7436.

[14] T. Kim, H. Shim, and D. D. Cho, "Distributed Luenberger observer design," in *Proc. IEEE Conf. Decis. Control*, 2016, pp. 6928–6933.

[15] A. Mitra and S. Sundaram, "Distributed observers for LTI systems," *IEEE Trans. Autom. Control*, vol. 63, no. 11, pp. 3689–3704, 2018.

[16] W. Han, H. L. Trentelman, Z. Wang, and Y. Shen, "A simple approach to distributed observer design for linear systems," *IEEE Trans. Autom. Control*, vol. 64, no. 1, pp. 329–336, 2019.

[17] T. Kim, C. Lee, and H. Shim, "Completely decentralized design of distributed observer for linear systems," *IEEE Trans. Autom. Control*, vol. 65, no. 11, pp. 4664–4678, 2020.

[18] S. Khoo and L. Xie, "Robust finite-time consensus tracking algorithm for multirobot systems," *IEEE/ASME Trans. Mechatron.*, vol. 14, no. 2, pp. 219–228, 2009.

[19] T. Basar and G. J. Olsder, *Dynamic noncooperative game theory: second edition*, ser. Classics in Applied Mathematics. SIAM, 1999.

[20] T.-T. Feng and E. K.-W. Chu, "Newton's method for coupled continuous-time algebraic Riccati equations," *J. Appl. Math. Comput.*, vol. 70, no. 2, pp. 1023–1042, April 2024.

[21] H. Abou-Kandil, G. Freiling, V. Ionescu, and G. Jank, *Matrix Riccati equations in control and systems theory*. Birkhäuser, 2003.

[22] S. Wang and M. Guay, "Distributed state estimation for jointly observable linear systems over time-varying networks," *Automatica*, vol. 163, p. 111564, 2024.

[23] H. Basu and R. G. Sanfelice, "Distributed state estimation of jointly observable linear systems under directed switching networks," in *Proc. IEEE Conf. Decis. Control*, 2024, pp. 2232–2237.

[24] H. K. Khalil, *Nonlinear systems*, 3rd ed. Upper Saddle River, NJ: Prentice Hall, 2002.