Saturated RISE Controllers with Exponential Stability Guarantees: A Projected Dynamical Systems Approach

Omkar Sudhir Patil, Rushikesh Kamalapurkar, and Warren E. Dixon

Abstract—**The Robust Integral of the Sign of the Error (RISE) control approach results in a powerful continuous controller that yields exponential tracking error convergence despite the presence of time-varying and state-dependent disturbances. However, designing the RISE controller to yield exponential tracking error convergence in the presence of actuator saturation has been an open problem. Although there are existing results that provide a saturation scheme for RISE controllers, these results only guarantee asymptotic tracking error convergence using a Lyapunov-based analysis. In this paper, a new design strategy is developed using a projection algorithm and auxiliary filters to yield exponential tracking error convergence. This new strategy does not employ trigonometric or hyperbolic saturation functions inherent to previous saturated (or amplitude limited) controllers. As a result, a Lyapunov-based analysis can be constructed that yields exponential convergence of the tracking errors. Comparative simulation results demonstrate the performance of the developed method in comparison with a baseline controller. The developed method can operate at a lower saturation limit than the baseline method while maintaining stability and achieving exponential tracking error convergence.**

I. INTRODUCTION

The family of Robust Integral of the Sign of the Error (RISE) controllers provide a powerful continuous control method that yields asymptotic tracking error convergence despite the presence of time-varying and state-dependent disturbances [1]–[11]. In [12], the tracking error convergence is also shown to be uniform and exponential, and the exponential tracking result is shown to also hold for systems involving state delays in [13]. The results in [12] and [13] enable the stability analysis using a specialized function called the P-function, which is used in the candidate Lyapunov function to prove exponential tracking.

Although the results in [12] and [13] provide exponential tracking error convergence, the development and analysis does not account for actuator saturation, which is a common issue in control systems. Traditional control methods [14]–[16] use passivity-based design techniques leveraging the small-gain theorem to compensate for the nonlinearities introduced by the saturation constraints. However, due to the lack of a high-frequency component like sliding-mode or RISE, these passivity-based design techniques typically do not achieve exponential stability guarantees in the presence of external disturbances. Therefore, motivation exists to develop continuous control methods that can achieve exponential stability despite the saturation constraints and external disturbances. To implement RISE controllers in the presence of actuator saturation, a dynamic saturation scheme is provided in [17], and a Lyapunov-based stability analysis is provided to yield asymptotic tracking guarantees despite the saturation. However, the stability analysis in [17] only guarantees asymptotic convergence of the tracking error without guaranteeing exponential convergence. The novel P-function introduced in [12] cannot be easily applied for the result in [17] because the candidate Lyapunov function (V_L) in [17] involves a combination of linear, quadratic, logarithmic, and hyperbolic functions of the state, e.g., $\ln(\cosh(e_1))$ term, where e_1 denotes the tracking error. Consequently, there are mathematical challenges in extending the analytical development in [12] for the controller in [17] to yield an inequality of the form $\dot{V}_L \leq -\lambda V_L$ which is essential in [12] to yield exponential convergence with some constant rate of convergence $\lambda \in \mathbb{R}_{>0}$. Moreover, the saturation mechanism in [17] involves an integrator of the form

$$\begin{aligned} u &= \gamma \tanh(v) \\ \dot{v} &= \frac{1}{\gamma}\cosh^2(v)(\mu), \end{aligned} \tag{1}$$

where γ is the saturation bound and μ is a nominal input to the integrator. This mechanism then yields $\dot{u} = \mu$, provided v does not escape in finite time so that $\cosh^2(v)$ and $\operatorname{sech}^2(v)$ can cancel. The issue with saturation mechanisms of the form in (1) is, if v is bounded, they generate solutions identical to the unsaturated integrator

$$\dot{u} = \mu. \tag{2}$$

This is only possible if the solutions to (2) never reach the saturation limit, in which case it might be preferable to use (2) instead of (1). Otherwise, a contradiction results, leading to the conclusion that v must be unbounded when

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the saturation limit is reached. Although the result in [17] establishes the boundedness of u, there are difficulties in finding conditions under which v does not escape in finite time. This is a common problem with approaches that pass the output of an integrator through saturating functions like tanh. Instead, it is preferable to modify the input to the integrator such that the solutions to the integrator would satisfy the saturation bound.

In this paper, we present a new design and stability analysis using a projection operator-based approach to design a saturated RISE controller with exponential stability guarantees. By projecting the input to an integrator on the tangent cone to the saturation region, the projection operator constrains the resulting integral to the saturation region. As a result, the control input always satisfies the saturation constraints. Although the projection operator achieves input saturation, it is unclear if the projection-based approach would preserve the exponential stability guarantees established in [12] and [13]. To answer this question, we first prove an extension of an important property of projection operators in [18, Lemma E.1.IV] for closed convex sets involving nonsmooth boundaries such as component-wise saturation. Based on this property, we derive conditions under which exponential tracking error convergence can be achieved using the modified RISE controller, despite the projection.

Modifying a RISE controller using projection is difficult because the standard RISE controller contains a state-derivative term in the integrator, which is difficult to separate from the integrator when a projection operator is applied. To overcome this challenge, we employ an auxiliary filter to avoid the use of a state-derivative term in the projected integrator. Subsequently, we construct a filtered tracking error based on the auxiliary filter that can be represented as the difference between the uncertainty and the control input. This representation enables us to use the properties of the projection operator in the Lyapunov-based stability analysis to guarantee exponential tracking error convergence, provided the system uncertainty lies within a compact convex set. Unlike the method presented in [17], our solution guarantees the boundedness of all closed-loop signals and exponential tracking error convergence, provided the system is stabilizable with a saturated control input. Comparative simulation results are provided to demonstrate the performance of the developed method, and the results are compared with the method in [17]. The developed method can operate at a lower saturation limit than the baseline method while maintaining stability and achieving exponential tracking error convergence. Upon selecting a higher saturation limit for the baseline method to avoid instability, and selecting the parameters for both methods to yield approximately the same root mean squared (RMS) control effort, the developed method is able to provide 20% reduction in the RMS control effort.

II. Notation and Preliminaries

The Lebesgue measure on $\mathbb{R}^n$ is denoted by m. The notation $\mathcal{C}^m$ denotes the space of continuous functions with continuous first m derivatives. The p-norm is denoted by $\|\cdot\|_p$, and $\|\cdot\|$ denotes the 2-norm. Given some sets A and B, a set-valued map F from A to subsets of B is denoted by $F : A \rightrightarrows B$. The notation $\overline{\mathrm{co}}A$ denotes the closed convex hull of the set A. The notation $\mathbb{B}(x, \delta)$, for $x \in \mathbb{R}^n$ and $\delta > 0$, is used to denote the set $\{y \in \mathbb{R}^n : \|x - y\| < \delta\}$. Consider a Lebesgue measurable and locally essentially bounded function $h : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$. The Filippov regularization of h at $(y, t) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$ is defined as the intersection of convex closures of values attained by h in every neighborhood of y omitting sets of measure zero, i.e.,

$$K[h](y,t) \triangleq \bigcap_{\delta>0} \bigcap_{\mathrm{m}S_m=0} \overline{\mathrm{co}}\, h\left(\mathbb{B}(y,\delta) \setminus \mathbb{S}, t\right),$$

where $\bigcap_{\mathrm{m}\mathbb{S}=0}$ denotes the intersection over all sets $\mathbb{S} \subset \mathbb{R}^n$ of Lebesgue measure zero [19, Equation 2b]. Additionally, given any sets $A, B \subset \mathbb{R}$, the notation $A \leq B$ is used to state $a \leq b$ for all $a \in A$ and $b \in B$. The notation $\overset{a.a.t.}{(\cdot)}$ implies that the relation $(\cdot)$ holds for almost all $t \in \mathcal{I}$, given some interval $\mathcal{I}$. A function $y : \mathcal{I}_y \to \mathbb{R}^n$ is called a Filippov solution of $\dot{y} = h(y, t)$ on the interval $\mathcal{I}_y \subseteq \mathbb{R}_{\geq 0}$, if y is absolutely continuous on $\mathcal{I}_y$, and is a solution to the differential inclusion $\dot{y} \overset{a.a.t.}{\in} K[h](y, t)$. The gradient operator is denoted by ∇, and Clarke's generalized gradient for a locally Lipschitz function $V : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ at $(x, t) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$ is defined as the convex closure of its gradients in an arbitrarily small neighborhood of (x, t) while omitting sets of measure zero where it is not defined, i.e.,

$$\partial V(x,t) \triangleq \overline{\mathrm{co}}\{\lim \nabla V(x,t) | (x_i,t_i) \to (x,t), (x_i,t_i) \notin \Omega_V\},$$

where Ω_V denotes the set of measure zero wherever ∇V is not defined [20, Def. 2.2].

A. Projection Operator

Given a closed set $\Theta \subset \mathbb{R}^n$ and $\theta \in \Theta$, a vector v is a tangent vector to Θ at θ if there exist sequences $\theta_k \to \theta$ with $\theta_k \in \Theta$ for all $k \in \mathbb{Z}_{>0}$ and $\delta_k \to 0^+$ such that $\frac{\theta_k - \theta}{\delta_k} \to v$. The set of all tangent vectors to Θ at θ is called the tangent cone at θ and denoted by $T_\theta\Theta$ [21]. If the set-valued map $\theta \mapsto T_\theta\Theta$ is inner semicontinuous (i.e., $\liminf_{\hat{\theta}\to\theta} T_{\hat{\theta}}\Theta \supset T_\theta\Theta$ [22, Def. 5.4]), then the set Θ is termed Clarke regular (or tangentially regular) [21]. For the set Θ, the projection operator at $\theta \in \Theta$ for any given argument $\mu \in \mathbb{R}^n$ is defined as [21]

$$\mathrm{proj}_\Theta^\theta(\mu) \triangleq \underset{v \in T_\theta\Theta}{\arg\min} \|v - \mu\|^2.$$

In the following lemma, we establish an important property of projection operators which is used for ensuring exponential stability guarantees in this paper. This is essentially a generalization of [18, Lemma E.1.IV] to nonsmooth projected dynamic systems involving nonsmooth closed convex sets.

Lemma 1. *Given a closed convex Clarke regular set $\Theta \subset \mathbb{R}^n$ and any point $\theta^* \in \Theta$, the inequality*

$-(\theta^* - \theta)^T K\left[\mathrm{proj}_\Theta^\theta\right](\mu) \le -(\theta^* - \theta)^T \mu$ *holds for all* $\theta \in \Theta$ *and* $\mu \in \mathbb{R}^n$.

Proof: Because Θ is convex, the tangent cone $T_\theta\Theta$ is also convex. To establish this fact, recall that by the definition of $T_\theta\Theta$, for all $v_1, v_2 \in T_\theta\Theta$, there exist sequences $\theta_{1,k}, \theta_{2,k} \to \theta$ with $\theta_{1,k}, \theta_{2,k} \in \Theta$ for all $k \in \mathbb{Z}_{>0}$ and $\delta_k \to 0^+$ such that $\frac{\theta_{1,k}-\theta}{\delta_k} \to v_1$ and $\frac{\theta_{2,k}-\theta}{\delta_k} \to v_2$. Due to the convexity of Θ, $\alpha\theta_{1,k} + (1-\alpha)\theta_{2,k} \in \Theta$ for all $\alpha \in [0,1]$. Thus, constructing the series $\theta_{3,k} \triangleq \alpha\theta_{1,k} + (1-\alpha)\theta_{2,k} \in \Theta$ yields $\frac{\theta_{3,k}-\theta}{\delta_k} = \alpha\left(\frac{\theta_{1,k}-\theta}{\delta_k}\right) + (1-\alpha)\left(\frac{\theta_{2,k}-\theta}{\delta_k}\right) \to \alpha v_1 + (1-\alpha)v_2$ for all $\alpha \in [0,1]$; thus $\alpha v_1 + (1-\alpha)v_2 \in T_\theta\Theta$, implying that $T_\theta\Theta$ is convex.

For a given μ, let $g_\mu : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ be defined $g_\mu(v) \triangleq \|v - \mu\|^2$ for all $v \in \mathbb{R}^n$, and $v^* \triangleq \mathrm{proj}_\Theta^\theta(\mu) = \arg\min_{v \in T_\theta\Theta} g_\mu(v)$. Due to the convexity of $T_\theta\Theta$, the inequality $-\nabla g_\mu(v^*)^T(v - v^*) = (\mu - v^*)^T(v - v^*) \le 0$ holds for all $v \in T_\theta\Theta$ by the first-order optimality condition [22, Theorem 6.12]. Moreover, due to the convex cone property of $T_\theta\Theta$, for any $v_1, v_2 \in T_\theta\Theta$, the relation $\omega_1 v_1 + \omega_2 v_2 \in T_\theta\Theta$ holds for all $\omega_1, \omega_2 \in \mathbb{R}_{>0}$. Thus, since $v^* \in T_\theta\Theta$, selecting $v = w + v^*$ yields $(\mu - v^*)^T(v - v^*) = (\mu - v^*)^T w \le 0$ for all $w \in T_\theta\Theta$. From the definition of tangent cone and convexity of Θ, $\theta^* - \theta \in T_\theta\Theta$ for all $\theta^* \in \Theta$. This fact is shown by selecting $\theta_k = \delta_k\theta^* + (1 - \delta_k)\theta \in \Theta$, which yields $\lim_{\delta_k \to 0^+} \frac{\theta_k - \theta}{\delta_k} \to \theta^* - \theta$. Therefore, $(\mu - v^*)^T(\theta^* - \theta) \le 0$ for all $\theta^* \in \Theta$, which can be rewritten as $-(\theta^* - \theta)^T \mathrm{proj}_\Theta^\theta(\mu) \le -(\theta^* - \theta)^T \mu$ by recalling $v^* = \mathrm{proj}_\Theta^\theta(\mu)$. Due to the facts that the aforementioned inequality is linear in $\mathrm{proj}_\Theta^\theta(\mu)$ and $K[\cdot]$ is convex, the inequality $-(\theta^* - \theta)^T v \le -(\theta^* - \theta)^T \mu$ holds for all $v \in K\left[\mathrm{proj}_\Theta^\theta\right](\mu)$. Therefore, $-(\theta^* - \theta)^T K\left[\mathrm{proj}_\Theta^\theta\right](\mu) \le -(\theta^* - \theta)^T \mu$ holds for all $\theta \in \Theta$ and $\mu \in \mathbb{R}^n$. ■

The following lemma states the forward invariance properties of projected dynamic systems, which will be used for imposing saturation constraints on the control input.

Lemma 2. *(Forward Invariance) Consider a closed convex Clarke regular set* $\Theta \subset \mathbb{R}^n$ *and a continuous vector field* $F : \Theta \times \mathbb{R}_{\ge 0} \to \mathbb{R}^n$. *Then Filippov solutions to* $\dot{\theta} = \mathrm{proj}_\Theta^\theta(F(\theta, t))$ *initialized with* $\theta(0) \in \Theta$ *exist, and every such Filippov solution satisfies* $\theta(t) \in \Theta$ *for all* $t \in \mathbb{R}_{\ge 0}$, *if i) F is bounded or globally Lipschitz, or ii)* Θ *is bounded.*

Proof: Under the stated conditions on F and Θ, the result in [21, Corollary 4.3] guarantees existence and completeness of all Krasovskii solutions to $\dot{\theta} = \mathrm{proj}_\Theta^\theta(F(\theta, t))$ subjected to $\theta \in \Theta$. Notably, a Filippov solution is always a Krasovskii solution (see [23, Eq. (4) and (5)] for the differences in their definitions); thus, the completeness can also be concluded for Filippov solutions. ■

Although the projection operator is defined as a solution to a quadratic program, closed-form expressions can be obtained for various cases of practical importance. The following example derives a closed form of the projection operator for sets describing channel-wise saturation.

Example 1. Consider the set $\Theta = \{\theta = [\theta_1, \theta_2, \ldots, \theta_n]^T \in \mathbb{R}^n : \underline{\theta_i} \le \theta_i \le \overline{\theta_i}\}$,where $\underline{\theta_i}, \overline{\theta_i} \in \mathbb{R}$ denote the upper and lower bounds on θ_i for all $i \in \{1, \ldots, n\}$. Then the tangent cone $T_\theta\Theta$ is given by $T_\theta\Theta = \{v = [v_1, v_2, \ldots, v_n]^T \in \mathbb{R}^n : v_i \ge 0 \text{ if } \theta_i = \underline{\theta_i},\ v_i \le 0 \text{ if } \theta_i = \overline{\theta_i},\ v_i \in \mathbb{R} \text{ if } \underline{\theta_i} < \theta_i < \overline{\theta_i}\}$. Since the tangent cone is independently defined for each θ_i, the projection operator can be applied component wise. Thus, using the expression for $T_\theta\Theta$ and the definition of the projection operator, the projection $\mathrm{proj}_\Theta^\theta(\mu)$ is given by $\mathrm{proj}_\Theta^\theta(\mu)_i = \max(0, \mu_i)$ if $\theta_i = \underline{\theta_i}$, $\min(0, \mu_i)$ if $\theta_i = \overline{\theta_i}$, and μ_i if $\underline{\theta_i} < \theta_i < \overline{\theta_i}$, where μ_i and $\mathrm{proj}_\Theta^\theta(\mu)_i$ denote the i^{th} elements of μ and $\mathrm{proj}_\Theta^\theta(\mu)$, respectively.

The following example derives the traditional closed-form projection operator used in adaptive control [18, Appendix E] for sets with boundaries described by $\mathcal{C}^1$ functions, using the tangent cone-based definition of projection operation.

Example 2. Consider the set $\Theta = \{\theta \in \mathbb{R}^n : h(\theta) \le 0\}$, where $h : \mathbb{R}^n \to \mathbb{R}$ is $\mathcal{C}^1$. For points that lie in the interior of Θ, i.e., where $h(\theta) < 0$, the tangent cone is the entire space, $T_\theta\Theta = \mathbb{R}^n$; thus $\mathrm{proj}_\Theta^\theta(\mu) = \mu$ if $h(\theta) < 0$. For points on the boundary of Θ, i.e., where $h(\theta) = 0$, the tangent vector v must point into Θ, i.e., $\nabla h(\theta)^T v \le 0$; therefore, $T_\theta\Theta = \{v \in \mathbb{R}^n : \nabla h(\theta)^T v \le 0\}$ if $h(\theta) = 0$. For points on the boundary, $\mathrm{proj}_\Theta^\theta(\mu)$ is a solution to the optimization problem $\arg\min_v \|v - \mu\|^2$ s.t. $\nabla h(\theta)^T v \le 0$. This is a convex quadratic optimization problem with a linear constraint, which can be solved by introducing the Lagrange multiplier $\Lambda \in \mathbb{R}_{\ge 0}$, where the Lagrangian is $L(v, \lambda) = \|v - \mu\|^2 + \Lambda\nabla h(\theta)^T v$. To find the minimizer v^*, Karush-Kuhn-Tucker (KKT) conditions are imposed. By the stationarity condition, $\nabla_v L(v^*, \lambda) = 2(v^* - \mu) + \Lambda\nabla h(\theta) = 0$, which yields $v^* = \mu - \frac{\Lambda}{2}\nabla h(\theta)$. Then using the primal feasibility condition $\nabla h(\theta)^T v^* \le 0$ yields $\nabla h(\theta)^T\left(\mu - \frac{\Lambda}{2}\nabla h(\theta)\right) \le 0$, therefore $\Lambda \ge \frac{2\nabla h(\theta)^T \mu}{\|\nabla h(\theta)\|^2}$. Furthermore, imposing the dual feasibility condition $\Lambda \ge 0$ yields $\Lambda \ge \max\left(0, \frac{2\nabla h(\theta)^T \mu}{\|\nabla h(\theta)\|^2}\right)$. Finally, imposing the complementary slackness condition $\Lambda\nabla h(\theta)^T v^* \le 0$ yields $\Lambda\left(\nabla h(\theta)^T \mu - \frac{\Lambda}{2}\|\nabla h(\theta)\|^2\right) = 0$, which implies two cases. In the first case, $\Lambda = 0$, which implies $\Lambda = \mu$. In the second case, $\nabla h(\theta)^T \mu - \frac{\Lambda}{2}\|\nabla h(\theta)\|^2 = 0$, implying $\Lambda = \frac{2\nabla h(\theta)^T \mu}{\|\nabla h(\theta)\|^2}$. These cases together with the condition $\Lambda \ge \max\left(0, \frac{2\nabla h(\theta)^T \mu}{\|\nabla h(\theta)\|^2}\right)$ imply $\Lambda = \max\left(0, \frac{2\nabla h(\theta)^T \mu}{\|\nabla h(\theta)\|^2}\right)$. Substituting Λ back into v^* yields $v^* = \mu - \max\left(0, \frac{\nabla h(\theta)^T \mu}{\|\nabla h(\theta)\|^2}\right)\nabla h(\theta)$. Therefore, the projection operator is given by

$$\mathrm{proj}_\Theta^\theta(\mu) = \begin{cases} \mu, & h(\theta) < 0 \\ \mu - \max\left(0, \frac{\nabla h(\theta)^T \mu}{\|\nabla h(\theta)\|^2}\right)\nabla h(\theta), & h(\theta) = 0, \end{cases}$$

which is the same as the projection operator in [18, Appendix E].

III. Control Design

A. Problem Statement

Consider the nonlinear system

$$\ddot{x} = f(x, \dot{x}, t) + d(t) + u, \tag{3}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^n$ is the control input, $f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is a $\mathcal{C}^2$ uncertainty such that the mappings $t \mapsto f(x, \dot{x}, t)$, $t \mapsto \nabla f(x, \dot{x}, t)$, and $t \mapsto \nabla^2 f(x, \dot{x}, t)$ are uniformly bounded, and $d : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is a $\mathcal{C}^2$ disturbance such that d, $\dot{d}$, and $\ddot{d}$ are uniformly bounded. The control objective is to ensure that the tracking error $e_1 \triangleq x_d - x$ exponentially converges to zero while ensuring that u stays saturated within a given closed convex Clarke regular set $\Omega \subset \mathbb{R}^n$ containing the origin in its interior, where $x_d : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is a $\mathcal{C}^4$ reference trajectory such that $x_d, \dot{x}_d, \ddot{x}_d, \dddot{x}_d$, and $\ddddot{x}_d$ are uniformly bounded.

B. Control Development

Let $\mathcal{I} = [t_0, t_1)$ denote an interval of time during which solutions to the closed-loop error system in the subsequent development exist with some $t_0, t_1 \in \mathbb{R}_{\geq 0}$. To facilitate the control development without requiring $\ddot{x}$ measurements, an auxiliary term $e_f \in \mathbb{R}^n$ is designed as a solution to the filter

$$\dot{e}_f = -\gamma_1 e_2 - \gamma_2 e_f, \tag{4}$$

where $\gamma_1, \gamma_2 \in \mathbb{R}_{>0}$ are user-defined constants. Additionally, let the auxiliary errors $e_2, r \in \mathbb{R}^n$ be defined as

$$e_2 = \dot{e}_1 + \alpha_1 e_1 + e_f, \tag{5}$$

$$r = \dot{e}_2 + \alpha_2 e_2, \tag{6}$$

where $\alpha_1, \alpha_2 \in \mathbb{R}_{>0}$. To ensure the control input stays saturated within Ω, we design the control input as $u = \hat{S}$ using a projection-based adaptive update law, where the projection operator is used to ensure $\hat{S} \in \Omega$. The control input u is designed as $u = \hat{S}$, where $\hat{S} \in \mathbb{R}^n$ is designed as a Filippov solution to

$$\dot{\hat{S}} = \mathrm{proj}_{\Omega}^{\hat{S}} \left(\alpha_1^3 e_1 + k_1 e_2 + k_2 e_f + \beta \mathrm{sgn}(e_2) \right), \tag{7}$$

where $k_1, k_2 \in \mathbb{R}_{>0}$ are constant control gains that are designed subsequently. Taking the time-derivative of e_2 and substituting (3)-(5) into the resultant expression yields

$$\begin{aligned} \dot{e}_2 &= \ddot{x}_d - f(x, \dot{x}, t) - d(t) - u - \alpha_1^2 e_1 \\ &\quad + (\alpha_1 - \gamma_1) e_2 - (\alpha_1 + \gamma_2) e_f. \end{aligned} \tag{8}$$

Substituting (8) into (6) yields

$$\begin{aligned} r &= \ddot{x}_d - f(x, \dot{x}, t) - d(t) - \hat{S} \\ &\quad - \alpha_1^2 e_1 - m_1 e_2 - m_2 e_f, \end{aligned} \tag{9}$$

where $m_1, m_2 \in \mathbb{R}_{>0}$ are constants defined as $m_1 \triangleq \gamma_1 - \alpha_1 - \alpha_2$ and $m_2 \triangleq \alpha_1 + \gamma_2$. Additionally, to facilitate the subsequent analysis, let $z \in \mathbb{R}^{4n}$ be defined as $z \triangleq \begin{bmatrix} e_1^T & e_2^T & e_f^T & r^T \end{bmatrix}^T$ and $S(z, t) \in \mathbb{R}^n$ be defined as

$$S(z, t) \triangleq \ddot{x}_d - f(x, \dot{x}, t) - d(t) - \alpha_1^2 e_1 - m_1 e_2 - m_2 e_f. \tag{10}$$

Based on (9) and (10), r can be rewritten as

$$r = S(z, t) - \hat{S}. \tag{11}$$

This representation of r is useful since it allows for the use of Lemma 1 in the subsequent stability analysis to achieve the exponential stability result. The following assumption is made to ensure the existence of a region of attraction in the subsequent analysis.

Assumption 1. There exists a closed set $\mathcal{D} \subset \mathbb{R}^{4n}$ such that $z \in \mathcal{D}$ implies $S(z, t) \in \Omega$ for all $t \in \mathbb{R}_{\geq 0}$.

Remark 1. Assumption 1 provides a sufficient feasibility condition on the region of attraction on how much disturbance or drift can be tolerated and how much acceleration the desired trajectory can involve for a given saturation bound on the control input. To illustrate this fact, let $\bar{u} \triangleq \sup R_u$ s.t. $\{\varsigma \in \mathbb{R}^n : \|\varsigma\| \leq R_u\} \subseteq \Omega$ denote the saturation limit, $\bar{D} = \sup_{t \geq 0} \|\ddot{x}_d - f(x_d, \dot{x}_d, t) - d(t)\|$ denote the disturbance bound, and $\rho_f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a strictly increasing function satisfying $\sup_{t \geq 0} \|f(x_d, \dot{x}_d, t) - f(x, \dot{x}, t)\| \leq \rho_f(\|z\|) \|z\|$, the existence of which follows from the mean value theorem-based inequality in [24, Lemma 5] and the fact that $t \mapsto \nabla f(x, \dot{x}, t)$ is uniformly bounded by definition. Provided $\bar{u} > \bar{D}$, the set $\mathcal{D}$ can be sufficiently computed as $\mathcal{D} = \{z \in \mathbb{R}^{4n} : \rho_f(\|z\|) \|z\| + (\alpha_1^2 - \alpha_2 + \gamma_1 + \gamma_2) \|z\| \leq \bar{u} - \bar{D}\}$. This explicit restriction makes it clear that arbitrary disturbances and arbitrary desired trajectories cannot be globally tracked using a saturated controller, unlike the result in [17].

Taking the time-derivative of both sides in (9), and substituting (4), (7), and (8) into the resultant expression yields that r is a Filippov solution to

$$\begin{aligned} \dot{r} &= N_B + \widetilde{N} - e_2 - m_1 r + \left(m_2 \gamma_1 - m_1 \alpha_2 - \alpha_1^2 \right) e_2 \\ &\quad - \mathrm{proj}_{\Omega}^{\hat{S}} \left(\alpha_1^3 e_1 + k_1 e_2 + k_2 e_f + \beta \mathrm{sgn}(e_2) \right) \\ &\quad + \left(m_2 \gamma_2 + \alpha_1^2 \right) e_f + \alpha_1^3 e_1, \end{aligned} \tag{12}$$

where $N_B \triangleq \dddot{x}_d - \frac{d}{dt} f(x_d, \dot{x}_d, t) - \dot{d}(t)$ and $\widetilde{N} \triangleq \frac{d}{dt} f(x_d, \dot{x}_d, t) - \frac{d}{dt} f(x, \dot{x}, t) + e_2$. Due to the facts that f is $\mathcal{C}^2$ and d, $\dot{d}$, $\ddot{d}$, $x_d, \dot{x}_d, \ddot{x}_d$, $\dddot{x}_d$, $\ddddot{x}_d$, $t \mapsto f(x, \dot{x}, t)$, $t \mapsto \nabla f(x, \dot{x}, t)$, and $t \mapsto \nabla^2 f(x, \dot{x}, t)$ are uniformly bounded, it follows that there exist constants $\chi_1, \chi_2 \in \mathbb{R}_{>0}$ such that

$$\|N_B\| \leq \chi_1 \tag{13}$$

and

$$\left\| \dot{N}_B \right\| \leq \chi_2. \tag{14}$$

Since $t \mapsto \nabla^2 f(x, \dot{x}, t)$ is uniformly bounded, the term $\widetilde{N}$ in (12) can be bounded using the mean-value theorem-based inequality in [24, Lemma 5] as

$$\left\|\widetilde{N}\right\| \leq \rho\left(\|z\|\right)\|z\|, \tag{15}$$

where $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a positive strictly increasing function.

Remark 2. The purpose of designing the auxiliary filter in (4) is to enable the appearance of the term $-m_1 r$ in (12) without having to use r (which is unknown because $\ddot{x}$ is unknown) in (7). The presence of $-m_1 r$ in (12) enables the exponential convergence result in the subsequent Lyapunov-based stability analysis by contributing a $-m_1\|r\|^2$ term.

The following section provides a Lyapunov-based stability analysis to show exponential tracking error convergence guarantees with the developed controller.

IV. Stability Analysis

The stability analysis for RISE controllers typically involves the use of an auxiliary P-function [2]. In [12], a unique P-function was introduced to show exponential tracking error convergence. Similarly, we introduce the P-function $P : \mathcal{I} \to \mathbb{R}_{\geq 0}$ defined as

$$\begin{aligned} P \;&\triangleq\; \beta\|e_2\|_1 - e_2^T N_B + \mathrm{e}^{-\lambda_P t} * \left(e_2^T \dot{N}_B\right) \\ &\quad + \mathrm{e}^{-\lambda_P t} * \left(\left(\alpha_2 - \lambda_P\right)\left(\beta\|e_2\|_1 - e_2^T N_B\right)\right), \end{aligned} \tag{16}$$

where $\lambda_P \in \mathbb{R}_{>0}$ is a user-selected constant, and the notation '$*$' denotes the convolutional integral $\mathrm{e}^{-\lambda_P t} * q = \int_{t_0}^{t} \mathrm{e}^{-\lambda_P(t-\sigma)} q(\sigma) d\sigma$, for any given $q : [t_0, \infty) \to \mathbb{R}$. Based on the Leibniz rule, the following property of convolutional integrals is obtained: $\frac{d}{dt}\left(\int_{t_0}^{t} \mathrm{e}^{-\lambda_P(t-\sigma)} q(\sigma) d\sigma\right) = q(t) - \lambda_P \int_{t_0}^{t} \mathrm{e}^{-\lambda_P(t-\sigma)} q(\sigma) d\sigma$. Therefore, $\frac{d}{dt}\left(\mathrm{e}^{-\lambda_P t} * q\right) = q(t) - \lambda_P \mathrm{e}^{-\lambda_P t} * q$. Since $t \mapsto e_2(t)$ is absolutely continuous and $\|\cdot\|_1$ is globally Lipschitz, the mapping $t \mapsto \|e_2(t)\|_1$ is differentiable for almost all time. Therefore, using the chain rule in [20, Theorem 2.2] yields $\frac{d}{dt}(\|e_2\|_1) \overset{a.a.t.}{\in} \dot{e}_2^T K[\mathrm{sgn}](e_2)$. Taking the time-derivative of (16), using Leibniz's rule, and substituting (6) and (16) into the resulting time-derivative yields

$$\dot{P} \overset{a.a.t.}{\in} -\lambda_P P + r^T \beta K[\mathrm{sgn}](e_2) - r^T N_B. \tag{17}$$

Moreover, provided the gain conditions

$$\alpha_2 > \lambda_P \tag{18}$$

and

$$\beta > \chi_1 + \frac{\chi_2}{\alpha_2 - \lambda_P} \tag{19}$$

are satisfied, it follows from the bounds in (13) and (14) that $\beta\|e_2\|_1 - e_2^T N_B \geq 0$ and $\left(\alpha_2 - \lambda_P\right)\left(\beta\|e_2\|_1 - e_2^T N_B\right) + e_2^T \dot{N}_B \geq 0$. Additionally, note that the convolution integrals of positive functions are positive, because for any given positive function $q : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, it follows that $\mathrm{e}^{-\lambda_P t} * q = \int_{t_0}^{t} \mathrm{e}^{-\lambda_P(t-\sigma)} q(\sigma) d\sigma \geq 0$. Therefore, examining the expression in (16) yields that $P \geq 0$, $\forall t \in \mathcal{I}$.

To state the main result of this paper, the following definitions are introduced. Let $W : \mathbb{R}^{4n} \to \mathbb{R}_{\geq 0}$ be defined as

$$W(\sigma) \;\triangleq\; \sqrt{\|\sigma\|^2 + 2\left(\beta + \chi_1\right)\|\sigma\|_1},\ \forall \sigma \in \mathbb{R}^{4n}. \tag{20}$$

Additionally, let $k_{\min} \in \mathbb{R}_{>0}$ be a constant gain defined as $k_{\min} \triangleq \min\{\alpha_1 - 1, \alpha_2 - 1, \gamma_2 - \frac{\gamma_1^2+1}{2}, m_1, \frac{\lambda_P}{2}\}$, $\lambda_V \in \mathbb{R}_{>0}$ be the desired rate of convergence, and the set $\mathcal{B} \subset \mathbb{R}^{4n}$ be defined as $\mathcal{B} \triangleq \{\sigma \in \mathbb{R}^{4n} : \rho\left(W\left(\sigma\right)\right) \leq k_{\min} - \lambda_V\}$.

Theorem 1. *All solutions to (4), (5), (8), and (12) with* $z(t_0) \in \mathcal{B}$ *satisfy* $\|z(t)\| \leq W(z(t_0))\mathrm{e}^{-\lambda_V(t-t_0)}$, $\forall t \in [t_0, \infty)$*, provided that the gains* $\alpha_1, \alpha_2, \gamma_1, \gamma_2$, β *and* λ_P *are selected according to the gain conditions in (18) and (19) and to ensure that* $\mathcal{B} \subset \mathcal{D}$*, the gains* k_1 *and* k_2 *are selected as*

$$k_1 \;=\; m_2\gamma_1 - m_1\alpha_2 - \alpha_1^2, \tag{21}$$

and

$$k_2 \;=\; m_2\gamma_2 + \alpha_1^2. \tag{22}$$

Proof: Let $\psi : \mathcal{I} \to \mathbb{R}^{4n+1}$ be defined as $\psi(t) \triangleq \begin{bmatrix} z^T(t) & P(t) \end{bmatrix}^T$, and $G : \mathbb{R}^{4n+1} \times \mathbb{R}_{\geq 0} \rightrightarrows \mathbb{R}^{4n+1}$ denote the set-valued map

$$G(\psi, t) \triangleq \begin{bmatrix} e_2 - \alpha_1 e_1 - e_f \\ r - \alpha_2 e_2 \\ -\gamma_1 e_2 - \gamma_2 e_f \\ \left(\begin{array}{c} N_B + \widetilde{N} - e_2 - m_1 r + \alpha_1^3 e_1 \\ -\beta K\left[\mathrm{proj}_{\Omega}^{\hat{S}}\right]\mathrm{sgn}(e_2) \\ -K\left[\mathrm{proj}_{\Omega}^{\hat{S}}\right]\left(\alpha_1^3 e_1 + k_1 e_2 + k_2 e_f\right) \\ +\left(m_2\gamma_1 - m_1\alpha_2 - \alpha_1^2\right)e_2 + \left(m_2\gamma_2 + \alpha_1^2\right)e_f \end{array}\right) \\ -\lambda_P P - r^T N_B + r^T \beta K[\mathrm{sgn}](e_2) \end{bmatrix}.$$

Then using (4), (5), (8), (12), and (17) yields $\dot{\psi} \overset{a.a.t.}{\in} G(\psi, t)$. Consider the function $V_L : \mathbb{R}^{4n+1} \to \mathbb{R}_{\geq 0}$ defined as

$$V_L\left(\psi\right) \triangleq \frac{1}{2}e_1^T e_1 + \frac{1}{2}e_2^T e_2 + \frac{1}{2}e_f^T e_f + \frac{1}{2}r^T r + P. \tag{23}$$

Based on the chain rule for differential inclusions in [20, Theorem 2.2], the time-derivative of V_L along the trajectories $t \mapsto \psi(t)$ exists for almost all time, and satisfies $\dot{V}_L(\psi, t) \overset{a.a.t.}{\in} \dot{\tilde{V}}_L(\psi, t)$, where the set $\dot{\tilde{V}}_L(\psi, t)$ is defined as $\dot{\tilde{V}}_L(\psi, t) \triangleq \bigcap_{\xi \in \partial V_L(\psi)} \xi^T G(\psi, t)$, where $\partial V_L(\psi)$ denotes Clarke's generalized gradient. Since the gradient of $V_L(\psi)$, i.e., $\nabla V_L(\psi)$, exists and is continuous for all $\psi \in \mathbb{R}^{4n+1}$,

$\partial V_L = \{\nabla V_L\}$. Therefore, $\dot{\tilde{V}}_L(\psi,t) = \nabla V_L^T G(\psi,t) = \begin{bmatrix} z^T & 1 \end{bmatrix} G(\psi,t)$. Evaluating $\dot{\tilde{V}}_L(\psi,t)$ yields

$$\begin{aligned}
\dot{\tilde{V}}_L(\psi,t) &= e_1^T\left(e_2-\alpha_1 e_1-e_f\right)+e_2^T\left(r-\alpha_2 e_2\right)\\
&\quad +e_f^T\left(-\gamma_1 e_2-\gamma_2 e_f\right)+r^T\left(N_B+\widetilde{N}-e_2\right)\\
&\quad -r^T K\left[\operatorname{proj}_\Omega^{\hat{S}}\right]\left(\alpha_1^3 e_1+k_1 e_2+k_2 e_f\right)\\
&\quad +r^T\left(-m_1 r+\left(m_2\gamma_1-m_1\alpha_2-\alpha_1^2\right)e_2\right)\\
&\quad +\left(m_2\gamma_2+\alpha_1^2\right)r^T e_f+\alpha_1^3 r^T e_1\\
&\quad -r^T\beta K\left[\operatorname{proj}_\Omega^{\hat{S}}\right]\operatorname{sgn}(e_2)\\
&\quad -\lambda_P P-r^T N_B+r^T\beta K[\operatorname{sgn}](e_2). \qquad (24)
\end{aligned}$$

Using Young's inequality yields $\gamma_1 e_f^T e_2 \leq \frac{1}{2}\left\|e_2\right\|^2+\frac{\gamma_1^2}{2}\left\|e_f\right\|^2$, $e_1^T e_2<\frac{1}{2}\left\|e_1\right\|^2+\frac{1}{2}\left\|e_2\right\|^2$, and $e_1^T e_f<\frac{1}{2}\left\|e_1\right\|^2+\frac{1}{2}\left\|e_f\right\|^2$. Therefore,

$$\begin{aligned}
\dot{\tilde{V}}_L(\psi,t) &\overset{a.a.t.}{\leq} -\left(\alpha_1-1\right)\left\|e_1\right\|^2-\left(\alpha_2-1\right)\left\|e_2\right\|^2\\
&\quad -\left(\gamma_2-\frac{\gamma_1^2+1}{2}\right)\left\|e_f\right\|^2-m_1\|r\|^2\\
&\quad +r^T\left(N_B+\widetilde{N}+\alpha_1^3 e_1\right)\\
&\quad -r^T K\left[\operatorname{proj}_\Omega^{\hat{S}}\right]\left(\alpha_1^3 e_1+k_1 e_2+k_2 e_f\right)\\
&\quad -r^T\beta K\left[\operatorname{proj}_\Omega^{\hat{S}}\right]\operatorname{sgn}(e_2)\\
&\quad +r^T\left(m_2\gamma_1-m_1\alpha_2-\alpha_1^2\right)e_2\\
&\quad +r^T\left(m_2\gamma_2+\alpha_1^2\right)e_f\\
&\quad -\lambda_P P-r^T N_B+r^T\beta K[\operatorname{sgn}](e_2). \quad (25)
\end{aligned}$$

Since $z(t_0)\in\mathcal{D}$, let $\mathcal{I}_\mathcal{D}\subset\mathcal{I}$ denote the time-interval over which $z(t)\in\mathcal{D}$ for all $t\in\mathcal{I}_\mathcal{D}$. Based on Assumption 1, $z\in\mathcal{D}$ implies $S(z,t)\in\Omega$; thus $S(z,t)\in\Omega$ for all $t\in\mathcal{I}_\mathcal{D}$. Therefore, using Lemma 1 and the fact $r=S-\hat{S}$ yields $-r^T K\left[\operatorname{proj}_\Omega^{\hat{S}}\right]\left(k_1 e_2+k_2 e_f+\beta\operatorname{sgn}(e_2)\right)\leq -r^T\left(k_1 e_2+k_2 e_f+\beta K[\operatorname{sgn}](e_2)\right)$ for all $t\in\mathcal{I}_\mathcal{D}$. As a result,

$$\begin{aligned}
\dot{\tilde{V}}_L(\psi,t) &\overset{a.a.t.}{\leq} -\left(\alpha_1-1\right)\left\|e_1\right\|^2-m_1\|r\|^2\\
&\quad -\left(\alpha_2-1\right)\left\|e_2\right\|^2\\
&\quad -\left(\gamma_2-\frac{\gamma_1^2+1}{2}\right)\left\|e_f\right\|^2\\
&\quad +r^T\left(\widetilde{N}-\left(\alpha_1^3 e_1+k_1 e_2+k_2 e_f\right)\right)\\
&\quad -r^T\beta K[\operatorname{sgn}](e_2)+\alpha_1^3 r^T e_1\\
&\quad +r^T\left(m_2\gamma_1-m_1\alpha_2-\alpha_1^2\right)e_2\\
&\quad +r^T\left(m_2\gamma_2+\alpha_1^2\right)e_f\\
&\quad -\lambda_P P+r^T\beta K[\operatorname{sgn}](e_2). \qquad (26)
\end{aligned}$$

Based on [12, Lemma 1], the set of time-instants where the term $r^T K[\operatorname{sgn}](e_2(\cdot))$ is set-valued has Lebesgue measure zero. As a result, $r^T K[\operatorname{sgn}](e_2(t))=\{r^T\operatorname{sgn}(e_2(t))\}$ for almost all $t\in\mathcal{I}_\mathcal{D}$. Therefore, substituting (21) and (22) into (26), and using (15) yields

$$\begin{aligned}
\dot{\tilde{V}}_L(\psi,t) &\overset{a.a.t.}{\leq} -\left(\alpha_1-1\right)\left\|e_1\right\|^2-m_1\|r\|^2\\
&\quad -\left(\alpha_2-1\right)\left\|e_2\right\|^2\\
&\quad -\left(\gamma_2-\frac{\gamma_1^2+1}{2}\right)\left\|e_f\right\|^2\\
&\quad +\rho(\|z\|)\|z\|^2-\lambda_P P\\
&\leq -\left(k_{\min}-\rho(\|z\|)\right)\|z\|^2-\lambda_P P\\
&\leq -2\left(k_{\min}-\rho(\|z\|)\right)\left(\frac{1}{2}\|z\|^2\right)-\lambda_P P,
\end{aligned}$$

where $k_{\min}$ is defined above the theorem statement. Since $\rho(\|z\|)\leq\rho\left(\sqrt{2V_L}\right)$, selecting $\lambda_P\geq 2\left(k_{\min}-\rho(\|z\|)\right)$ and recalling $\dot{V}_L(\psi,t)\overset{a.a.t.}{\in}\dot{\tilde{V}}_L(\psi,t)$ yields

$$\begin{aligned}
\dot{V}_L &\overset{a.a.t.}{\leq} -2\left(k_{\min}-\rho(\|z\|)\right)\left(\frac{1}{2}\|z\|^2+P\right)\\
&\leq -2\left(k_{\min}-\rho\left(\sqrt{2V_L}\right)\right)V_L.
\end{aligned}$$

Since $z(t_0)\in\mathcal{B}$, therefore $k_{\min}>\lambda_V+\rho\left(W\left(z\left(t_0\right)\right)\right)>\lambda_V+\rho\left(\sqrt{2V_L\left(\psi(t_0)\right)}\right)$. As a result,

$$\dot{V}_L\overset{a.a.t.}{\leq}-2\lambda_V V_L,\ \forall t\in\mathcal{I}_\mathcal{D}.$$

Based on the Comparison Principle [25, Lemma 4.4], it follows that

$$V_L(\psi(t))\leq V_L(\psi(t_0))\mathrm{e}^{-2\lambda_V(t-t_0)},\ \forall t\in\mathcal{I}_\mathcal{D}. \qquad (27)$$

Therefore, $k_{\min}\geq\lambda_V+\rho\left(\sqrt{2V_L\left(\psi(t_0)\right)}\right)\geq\lambda_V+\rho\left(\sqrt{2V_L\left(\psi(t)\right)}\right)$ for all $t\in\mathcal{I}_\mathcal{D}$, and which implies $z\in\mathcal{B}$ for all $t\in\mathcal{I}_\mathcal{D}$. Thus, if the gains are selected to ensure $\mathcal{B}\subseteq\mathcal{D}$, then z cannot escape $\mathcal{D}$, and therefore the time-interval $\mathcal{I}_\mathcal{D}$ can be extended into the entire interval of existence $\mathcal{I}$. Furthermore, using (27), $\|z(t)\|$ can further be upper-bounded as

$$\|z(t)\| \leq \sqrt{2V_L(\psi(t_0))}\mathrm{e}^{-\lambda_V(t-t_0)},\ \forall t\in\mathcal{I}. \quad (28)$$

Substituting $P(t_0)=\beta\left\|e_2(t_0)\right\|_1-e_2^T(t_0)N_B(t_0)$ yields $V_L(\psi(t_0))=\frac{1}{2}\left\|z(t_0)\right\|^2+\left(\beta\left\|e_2(t_0)\right\|_1-e_2^T(t_0)N_B(t_0)\right)$. Since the term $N_B(t_0)$ is bounded according to (13), using the facts that $\|e_2(t_0)\|\leq\|e_2(t_0)\|_1\leq\|z(t_0)\|_1$, it follows that

$$\begin{aligned}
\sqrt{2V_L(\psi(t_0))} &\leq \sqrt{\left\|z(t_0)\right\|^2+2\left(\beta+\chi_1\right)\left\|z(t_0)\right\|_1}\\
&= W(z(t_0)), \qquad (29)
\end{aligned}$$

where $W(\cdot)$ is defined in (20). Additionally, since $(\psi,t)\mapsto G(\psi,t)$ is a locally bounded mapping, and ψ is precompact (i.e., bounded over any interval $\mathcal{I}$) according to (28), invoking [26, Lemma 3.3 and Remark 3.4] rules out the possibility of solutions escaping in finite time. Therefore, $\mathcal{I}=[t_0,\infty)$. Thus, the exponential convergence in (28) holds for all $t\in[t_0,\infty)$. Therefore, substituting (29) into (28) yields

$$\|z(t)\| \leq W(z(t_0))\mathrm{e}^{-\lambda_V(t-t_0)},\ \forall t\in[t_0,\infty). \quad (30)$$

Because β, χ_1, and λ_V are independent of the initial time t_0 or initial condition $z(t_0)$, the exponential convergence is uniform [27]. Additionally, the convergence and boundedness of $\|z\|$ implies the convergence and boundedness of $\|e_1\|$, $\|e_2\|$, $\|e_f\|$, and $\|r\|$. Therefore, since $x_d, \dot{x}_d, \ddot{x}_d \in \mathcal{L}_\infty$, it can be concluded that $x, \dot{x}, \ddot{x} \in \mathcal{L}_\infty$ [28]. Thus, $f(x, \dot{x}, t)$ is bounded, and hence from (3), $u \in \mathcal{L}_\infty$. ■

Remark 3. To obviate the need to know the bounds χ_1 and χ_2 for the gain condition in (19), the dynamic gain scaling approach in [29] can be used to dynamically estimate the ideal gain $\beta^* \triangleq \chi_1 + \frac{\chi_2}{\alpha_2 - \lambda_P}$ using an adaptive estimate $\hat{\beta} \in \mathbb{R}$. Such an approach can be combined with the developed saturated RISE method by including an additional term $\frac{1}{2}\tilde{\beta}^2$ in V_L, where $\tilde{\beta} \triangleq \beta^* - \hat{\beta}$ denotes the gain estimation error. However, since $\tilde{\beta}$ is unknown, there are challenges in developing an adaptive update law $\dot{\hat{\beta}}$ that would yield a negative definite $\dot{V}_L$, thus restricting the result to asymptotic tracking error convergence, rather than the exponential convergence, if a dynamic gain scaling approach is used.

V. Simulation

An example system was simulated to provide an empirical demonstration of the developed controller, and the results are compared with the baseline saturated RISE controller developed in [17]. Specifically, the system in (3) is considered with

$$f(x, \dot{x}, t) = \begin{bmatrix} \cos(x_2) + 0.1\dot{x}_1 + 0.1x_2^2 \\ \sin(x_1) + 0.1\dot{x}_2 + 0.1x_1^2 \end{bmatrix}$$

and $d(t) = \begin{bmatrix} \sin(2t) & \cos(3t) \end{bmatrix}^T$, where $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$. The reference trajectory is $x_d(t) = 0.5 \begin{bmatrix} \sin(t) & \cos(t) \end{bmatrix}^T$. The baseline controller in [17] is given by

$$\begin{aligned} u &= \gamma_1 \tanh(\nu), \\ \dot{\nu} &= \cosh^2(\nu) \begin{pmatrix} \alpha_2 \tanh(e_2) + (\alpha_3 + \gamma_2) e_2 \\ +\beta \mathrm{sgn}(e_2) - \alpha_1 \mathrm{sech}^2(e_1) e_2 \end{pmatrix} \\ \dot{e}_f &= \cosh^2(e_f)(-\gamma_1 e_2 - \gamma_2 \tanh(e_f) + \tanh(e_1)) \\ e_2 &\triangleq \dot{e}_1 + \alpha_1 \tanh(e_1) + \tanh(e_f), \\ r &\triangleq \dot{e}_2 + \alpha_2 \tanh(e_2) + \alpha_3 e_2, \end{aligned}$$

with $\nu(t_0) = e_f(t_0) = 0$. The states are initialized as $x(0) = [-1, 2]^T$ and $\dot{x}(0) = [0, 0]^T$. The parameters used for the developed and baseline controllers are selected to yield approximately the same root mean squared (RMS) control input norm, and are listed in Table I.

The simulation is performed for 10 seconds. For the baseline method, selecting a saturation limit below 6 is found to cause instability in the simulation results. In contrast to the baseline method, the developed method results in tracking error convergence even with a saturation limit as low as 3. Figure 1 shows the comparative plots of the tracking error norm $\|e\|$ and the individual control inputs u_1 and u_2 with a channel-wise saturation limit of 3 with the developed method and 6 with the baseline method, and the corresponding RMS tracking error and control input norms are shown in Table II. As evident from the control input plots, the projection operator is able to saturate the control inputs at the desired saturation limit of 3 between 0-2 seconds. The tracking error converges in approximately 2.5 seconds with the developed method whereas the baseline takes 4.5 seconds. Despite operating at a lower saturation limit, the developed method is able to yield almost 1.8 times faster tracking error convergence than the baseline method. Additionally, the developed method produces a smoother control signal than the baseline in the transient state. This is likely because the baseline method involves $\cosh^2(\nu)\beta\mathrm{sgn}(e_2)$ term in $\dot{\nu}$, which introduces a larger high frequency components to the control input when ν becomes larger.

Table I
Controller Parameters

Parameter	Developed	Baseline
α_1	2	1
α_2	2	1
α_3	–	1
γ_1	10	6
γ_2	10	1
β	7	3

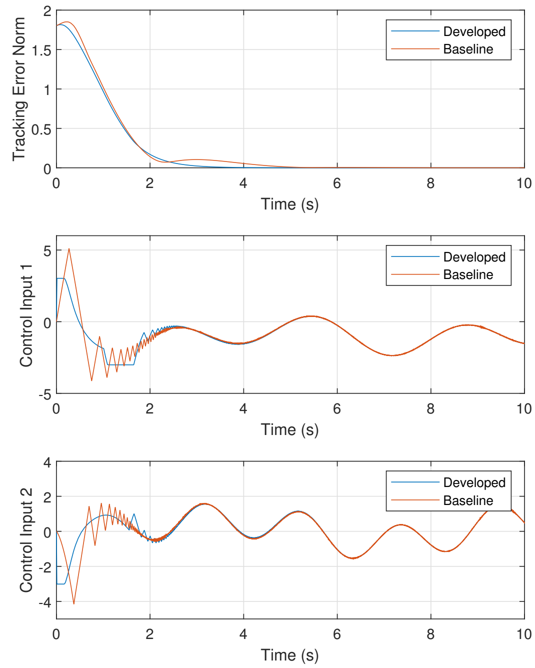

Figure 1. Plots of the tracking error norm and control inputs with the developed saturated RISE controller.

Table II
PERFORMANCE COMPARISON

Method	$\|e\|_{\mathrm{RMS}}$	$\|u\|_{\mathrm{RMS}}$
Developed	0.1407	1.5345
Baseline	0.1502	1.5333

VI. CONCLUSIONS

A saturated RISE controller is developed using a new design strategy that includes auxiliary filters and projection algorithm. This new strategy does not employ trigonometric or hyperbolic saturation functions inherent to previous saturated (or amplitude limited) controllers. As a result, we were able to construct a Lyapunov-based analysis that yields exponential convergence of the tracking errors. By leveraging properties of the projection operator, a Lyapunov-based analysis was used to show exponential tracking error convergence with the developed controller. Comparative simulation results are provided to demonstrate the performance of the developed method, and the results are compared with the method in [17]. The developed method can operate at a lower saturation limit than the baseline method. Upon selecting a higher saturation limit for the baseline method to avoid instability, and selecting the parameters for both methods to yield approximately the same root mean squared (RMS) control effort, the developed method is able to achieve approximately two fold faster tracking error convergence than the baseline method.

The stability analysis for robust nonlinear control methods such as RISE relies on conservative bounds on the uncertainty and reference trajectory. This conservativeness may restrict the sufficient gain conditions and region of attraction to more conservative values than what may potentially be possible with adaptive methods such as dynamic gain scaling. Future work may explore a dynamic gain scaling approach with the developed method that may potentially guarantee exponential stability.

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